1. **(8 points)** *Find an equation of the tangent line to the curve $y = \sqrt{25 - x^2}$ at $(3, 4)$.*

   Using the chain rule, if $y = \sqrt{u}$ and $u = 25 - x^2$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{u}}(-2x) = \frac{-x}{\sqrt{25-x^2}}$.

   Evaluated specifically at the point $x = 3$, one finds that $\frac{dy}{dx} = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$. Thus the slope of the tangent line to this curve at the point $(3, 4)$ is $\frac{-3}{4}$, and using point-slope form, the tangent line has equation

   $$y - 4 = \frac{-3}{4}(x - 3)$$

   which would, if written in slope-intercept form, be $y = \frac{-3}{4}x + \frac{25}{4}$.

2. **(8 points)** *The height in meters of a vertically-moving balloon at time t, measured in seconds, is given by the formula $s(t) = t^3 + t^2 - 8t + 4$.*

   (a) **(2 points)** *How quickly and in what direction is the balloon moving after 1 second?*

   The velocity function is $s'(t) = 3t^2 + 2t - 8$; thus the velocity after one second is $s'(1) = 3(1)^2 + 2(1) - 8 = -3$, which can be interpreted as a speed of 3 meters per second in the downwards (i.e. negative) direction.

   (b) **(2 points)** *What is the balloon's acceleration after 3 seconds?*

   Acceleration is given by $s''(t) = 6t + 2$, so after 3 seconds the acceleration is $s''(3) = 6(3)+2 = 20$, which implies acceleration of 14 meters per second per second *[sic]* upwards.

   (c) **(4 points)** *At what times is the balloon moving downwards?*

   The question here is equivalent to asking, "when is $s''(t)$ negative?" since downwards movement corresponds to negative velocity. From above, we have the function $s'(t) = 3t^2 + 2t - 8$, so the question is when $3t^2 + 2t - 8 < 0$; factoring this, we have $s'(t) = (3t - 4)(t + 2)$; so $s'(t)$ is negative when one of these factors is negative (i.e. $t < \frac{4}{3}$) and the other is positive (i.e. $t > -2$); thus the particule is moving to the right when t is between -2 and $\frac{4}{3}$. Depending on whether times less than zero are regarded as within the scope of our experiment at all ($t = 0$ was not afforded any special significance in this case, but it is sometimes regarded as the beginning time of any situation), one might restrict this range to t being between 0 and $\frac{4}{3}$.

3. **(8 points)** *Imre is twelve miles north of the Parliament, jogging southwards at six miles per hour; János is five miles to the east of Parliament, walking eastwards at three miles per hour.*

   (a) **(4 points)** *Is the distance between Imre and János increasing or decreasing, and at what rate?*

   Let us name various quantities which will change over time. First, we define time itself measured in hours as t, and name the physical quantities x, János's distance to the east of Parliament; y, Imre's distance to the north of Parliament; and s, the distance between them. Several relationships arise as a result of the problem statement: $x = 5$ at the time being investigated, $y = 12$ at the time being investigates, $\frac{dx}{dt} = 3$ (which is positive because János's distance from the Parliament is being increased by his movement), and $\frac{dy}{dt} = -6$ (which is negative because Imre's distance from the Parliament is being decreased by his movement). We also know that $s^2 = x^2 + y^2$ by the Pythagorean theorem, which will give us both the current distance between the two ($s = \sqrt{5^2 + 12^2} = 13$) and a relationship to be differentiated:

$$\begin{aligned}\frac{d}{dt}s^2 &= \frac{d}{dt}(x^2+y^2)\\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt}+2y\frac{dy}{dt}\\ \frac{ds}{dt} &= \frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2s}\\ &= \frac{2\cdot 5\cdot 3+2\cdot 12\cdot(-6)}{2\cdot 13} = \frac{-57}{13}\end{aligned}$$

Since this quantity is negative, the distance between the two is decreasing, at a rate of $\frac{57}{13}$ miles per hour.

(b) **(4 points)** *In an hour, will the distance between Imre and János increasing or decreasing, and at what rate?*

An hour later, the same relationships hold as above, but now $x = 5+3 = 8$, $y = 12-6 = 6$, and $s = \sqrt{8^2+6^2} = 10$, so we can calculate

$$\frac{ds}{dt} = \frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2s} = \frac{2\cdot 8\cdot 3+2\cdot 6\cdot(-6)}{2\cdot 10} = \frac{-6}{5}$$

so they are still becoming closer together, but at the much slower rate of $\frac{6}{5}$ miles per hour.

4. **(8 points)** *Differentiate* $\frac{\arctan t}{\ln(\sin t)}$ *with respect to* t*.*

This will require application of the quotient rule and the chain rule. We might anticipate needing to calculate $\frac{d}{dt}\ln(\sin t)$, and do so pre-emptively, or defer it until necessary. In either case, we would let $y = \ln u$, $u = \sin t$, and calculate $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dt} = \cos t$ to give the result from the chain rule that

$$\frac{d}{dt}\ln(\sin t) = \frac{dy}{du}\frac{du}{dt} = \frac{1}{u}\cos t = \frac{\cos t}{\sin t} = \cot t$$

Now we may invoke the quotient rule, armed with this knowledge:

$$\begin{aligned}\frac{d}{dt}\frac{\arctan t}{\ln(\sin t)} &= \frac{\ln(\sin t)\left(\frac{d}{dt}\arctan t\right) - \arctan t\left(\frac{d}{dt}\ln(\sin t)\right)}{(\ln(\sin t))^2}\\ &= \frac{\ln(\sin t)\left(\frac{1}{1+t^2}\right) - \arctan t\cot t}{(\ln(\sin t))^2}\end{aligned}$$

5. **(8 points)** *The* folium of Descartes *is a curve satisfying the equation* $x^3+y^3-5xy = 0$*.*

(a) **(6 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*

Taking the derivative of each side, and using rules as necessary:

$$\begin{aligned}
\frac{d}{dx}\left(x^3+y^3-5xy\right) &= \frac{d}{dx}0 \\
3x^2+\frac{d}{dx}y^3-5\left(y+x\frac{dy}{dx}\right) &= 0 \\
3x^2+\frac{dy}{dx}\frac{d}{dy}y^3-5\left(y+x\frac{dy}{dx}\right) &= 0 \\
3x^2+3y^2\frac{dy}{dx}-5y-5x\frac{dy}{dx} &= 0 \\
(3y^2-5x)\frac{dy}{dx} &= 5y-3x^2 \\
\frac{dy}{dx} &= \frac{5y-3x^2}{3y^2-5x}
\end{aligned}$$

(b) **(2 points)** *Identify conditions on x and y for the tangent lines to the folium to be horizontal and vertical (label which is which!).*

The tangent line is horizontal when $\frac{dy}{dx}=0$, which is the case when its numerator is zero. Thus, the criterion for a tangent line to be horizontal is $5y-3x^2=0$.

The tangent line is vertical when $\frac{dy}{dx}$ is undefined due to an infinite asymptote. This will occur when the denominator of $\frac{dy}{dx}$ is zero. Thus, the criterion for a tangent line to be vertical is $3y^2-5x=0$.

6. **(8 points)** *A collection of biological samples is taken from a $-200°F$ deep-freeze into a 50°F lab. After 10 minutes it has warmed up to -150°F.*

(a) **(4 points)** *Produce a function $T(t)$ modeling the samples' temperature t minutes after they are brought into the lab.*

We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 50°F, so our temperature model will be $T(t)=50+Ce^{kt}$; it remains only to find C and k to have a final model.

Since the samples have a temperature of -200°F immediately upon removal from the freezer, $T(0)=-200$. Evaluating the left side of this equation, we find that $50+Ce^0=-200$; thus $C=-250$.

Since the samples have a temperature of -150°F ten minutes later, we know that $T(10)=-150$. Expanding $T(10)$, we find that:

$$\begin{aligned}
50-250e^{k\cdot 10} &= -150 \\
-250e^{10k} &= -200 \\
e^{10k} &= \frac{4}{5} \\
10k &= \ln\frac{4}{5} \\
k &= \frac{\ln\frac{4}{5}}{10}
\end{aligned}$$

Assembling this value of k into our equation, we find that

$$T(t)=50-250e^{\frac{\ln\frac{4}{5}}{10}t}$$

(b) **(2 points)** *The samples will become biologically active when they reach 0°F. How long will it take for this to occur?*

Since the samples become active when $T(t) = 0$, we want to find the value of t satisfying that equation:

$$\begin{aligned} 50 - 250e^{\frac{\ln\frac{4}{5}}{10}t} &= 0 \\ -250e^{\frac{\ln\frac{4}{5}}{10}t} &= -50 \\ e^{\frac{\ln\frac{4}{5}}{10}t} &= \frac{1}{5} \\ \frac{\ln\frac{4}{5}}{10}t &= \ln\frac{1}{5} \\ t &= \frac{10\ln\frac{1}{5}}{\ln\frac{4}{5}} \approx 72 \text{ minutes} \end{aligned}$$

(c) **(2 points)** *How quickly are the samples' temperature changing ten minutes after being brought into the lab?*

The speed of the temperature change in ten minutes is $T'(10)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = -250\frac{\ln\frac{4}{5}}{10}e^{\frac{\ln\frac{4}{5}}{10}t}$$

so $T'(0) = -25(\ln\frac{4}{5})e^{\ln\frac{4}{5}} = -20\ln\frac{4}{5}$. This is approximately 4.46, signifying that the samples are warming (rising in temperature) by 4.46 degrees per minute.

7. **(8 points)** *Calculate* $\frac{d}{dx}\arcsin(x^2\tan x)$.

This is a chain rule problem, which will involve a product rule as a sub-problem. Letting $y = \arcsin u$ and $u = x^2\tan x$, we can calculate the derivatives $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and

$$\frac{du}{dx} = \left(\frac{d}{dx}x^2\right)\tan x + x^2\frac{d}{dx}\tan x = 2x\tan x + x^2\sec^2 x$$

so the original derivative is the product:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\left(2x\tan x + x^2\sec^2 x\right) = \frac{2x\tan x + x^2\sec^2 x}{\sqrt{1-(x^2\tan x)^2}}$$