1. **(8 points)** *Answer the following questions for the function $f(t) = -5\cos(3\pi t)$.*

   (a) **(3 points)** *What are its domain and range?*

   There are no domain-interrupting expressions in the definition if $f(t)$, so the domain is $(-\infty, \infty)$, or alternatively, all real t.

   The cosine function itself has range $[-1, 1]$, so multiplying by -5 stretches it to $[-5, 5]$.

   (b) **(2 points)** *Is it odd, even, both, or neither?*

   It is even, since the cosine function is even and the origin-symmetric transformations of stretching and flipping preserve that property.

   (c) **(3 points)** *What are its amplitude and period?*

   Its amplitude is 5, since it has been subjected to a vertical stretch of 5 (as well as a flip, whihc does not effect amplitude). It has been horizontally stretched by a factor of $\frac{1}{3\pi}$ (yielded by the multiplication inside the sine), so the sine function's period of 2π has been stretched to $2\pi \cdot \frac{1}{3\pi} = \frac{2}{3}$.

2. **(8 points)** *Given the function $g(x) = \frac{(x+1)(x^2+1)}{(x+1)(4x-2)(3x-1)}$, answer the following questions preparatory to sketching the functions.*

   (a) **(2 points)** What is the domain of the function?

   The denominator is zero when x is -1, $\frac{1}{2}$, or $\frac{1}{3}$; this prevents the function from having a value at these points. The domain may be given as restriction on x in the form $x \neq -1, \frac{1}{2}, \frac{1}{3}$, or as the interval notation $(-\infty, -1) \cup (-1, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

   (b) **(2 points)** *What are all the zeroes of the function?*

   The function is zero when the numerator is zero and the denominator is not zero (The $\frac{0}{0}$ form, as we have seen, is not necessarily a zero or an asymptote). Of the two factors in the numerator, $(x^2 + 1)$ is never zero, and $(x + 1)$ is zero when $x = -1$, so the only zero of the numerator is -1. Since -1 is also a zero of the denominator, the rational function as a whole has no zeroes.

   (c) **(2 points)** *What are all the vertical asymptotes of the function?*

   We already found the zeroes of the denominator in the first section of this problem; however, as we saw in the second section, $x = -1$ is not an asymptote. Thus the asymptotes are the other two undefined values, at $x = \frac{1}{2}$ and $x = \frac{1}{3}$.

   (d) **(2 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large or very negative values of x, $g(x)$ is approximately equal to the quotient of the highest-degree terms in the numerator and denominator. Multiplying out the highest-degree terms in each factor yields $\frac{x^3}{12x^3} = \frac{1}{12}$, so over the long term $g(x)$ approaches $\frac{1}{12}$. Thus, as $x \to \pm\infty$, $g(x) \to \frac{1}{12}$.

3. **(8 points)** Let $g(s) = \frac{-2s^2+5s-3}{s-1}$.

   (a) **(1 point)** *Find $\lim\limits_{s\to1} g(s)$.*

   Note that $g(s) = \frac{(-2s+3)(s-1)}{s-1}$. Thus, except at the point $s = 1$, $g(s) = -2s + 3$. Since the limit concerns the behavior not *at* $s = 1$ but in its vicinity, $\lim_{s\to1} \frac{-2s^2+5s-3}{s-1} = \lim_{s\to1} -2s + 3 = 1$.

(b) **(4 points)** *Using epsilon-delta methods, justify your result above.*

Given a value of ϵ, we constrain $g(s)$ to be within ϵ of 1, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned} 1-\epsilon < \frac{-2s^2+5s-3}{s-1} &< 1+\epsilon \\ 1-\epsilon < -2s+3 &< 1+\epsilon && \text{and } s \neq 1 \\ -2-\epsilon < -2s &< -2+\epsilon && \text{and } s \neq 1 \\ 1+\frac{\epsilon}{2} > s &> 1-\frac{\epsilon}{2} && \text{and } s \neq 1 \end{aligned}$$

So, since it is sufficient to reuqire x within $\frac{\epsilon}{2}$ of 1, we may establish δ to be $\frac{\epsilon}{2}$.

(c) **(3 points)** *State the mathematical definition of the expression* $\lim_{x\to a^-} f(x) = L$.

Given a (presumably very small) value $\epsilon < 0$, we can find a (presumably also small) value of δ such that if $a-\delta < x < a$, then $L-\epsilon < f(x) < L+\epsilon$.

4. **(8 points)** *Let* $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 3x+a & \text{if } 1 < x \le 4 \\ \sqrt{bx} & \text{if } x > 4 \end{cases}$.

What choices of a and b will make this function continuous?

Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 1$ and $x = 4$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 1$:

$$\begin{aligned} \lim_{x\to 1-} f(x) &= \lim_{x\to 1+} f(x) \\ \lim_{x\to 1-} x^2 &= \lim_{x\to 1+} 3x+a \\ 1^2 &= 3\cdot 1 + a \\ -2 &= a \end{aligned}$$

And likewise for $x = 4$:

$$\begin{aligned} \lim_{x\to 4-} f(x) &= \lim_{x\to 4+} f(x) \\ \lim_{x\to 4-} 3x+a &= \lim_{x\to 4+} \sqrt{bx} \\ 3\cdot 4 - 2 &= \sqrt{b\cdot 4} \\ 10 &= 2\sqrt{b} \\ 100 &= 4b \\ 25 &= b \end{aligned}$$

So our solution is to choose $a = -2$ and $b = 25$.

5. **(8 points)** *Let* $f(x) = x^2 - 4x$.

   (a) **(6 points)** *Using the difference quotient, find* $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2 - 4(x+h)] - (x^2 - 4x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - (x^2 - 4x)}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h} \\ &= \lim_{h\to 0} 2x + h - 4 \text{ justified since } h \neq 0 \\ &= 2x - 4 \end{aligned}$$

   (b) **(2 points)** *Find the equation of the tangent line to* $f(x)$ *at the point* $(-1, 5)$.

   We know this line must pass through $(-1, 5)$, with slope $f'(-1) = 2(-1) - 4 = -6$. Using point-slope form, we get

$$(y - 5) = -6(x + 1)$$

   which can also be expressed in slope-intercept form as $y = -6x - 1$.

6. **(8 points)** *Rex Tyler consumes 200mg of the drug Miraclo in the morning. 18% of the drug is eliminated from his system every hour thereafter.*

   (a) **(3 points)** *Create a function* $f(t)$ *to describe the quantity of Miraclo still in his body* t *hours after he takes the pill.*

   Since 18% of the drug is depleted per hour, 82% of it remains in his system after an hour; thus, in t hours, a proportion equal to 0.82^t of the original dose remains. Thus, $f(t) = 200\,(0.82^t)$.

   (b) **(5 points)** *In order to be able to sleep, he must have less than 50mg of the drug in his body. How soon will this occur?*

   We solve for t when $f(t) = 50$:

$$200\left(0.82^t\right) = 50 \; 0.82^t \qquad = \frac{50}{200} = \frac{1}{4}$$

$$t = \log_{0.82} \frac{1}{4} = \frac{\ln \frac{1}{4}}{\ln 0.82} \approx 7 \text{ hours}$$

Note: the described metabolic properties and dosages of Miraclo are quite similar to those of caffeine.

7. **(8 points)** *Evaluate the following limits; when a limit can not be evaluated, explain why or describe its behavior.*

   (a) **(2 points)** $\lim_{u \to 3+} \sqrt{9 - u^2}$.

   Since $9 - 3^2 = 0$, we must be careful in taking this limit: if the side on which we're looking at the function is one where $9 - u^2 > 0$, then the limit would be zero; but if $9 - u^2 < 0$, then the limit does not exist. We are considering u slightly larger than 3 in this case, so u^2 is slightly larger than 9, and $9 - u^2$ would thus be negative, making this limit nonexistent.

   (b) **(2 points)** $\lim_{\alpha \to 0} \frac{4\alpha^2 - 2\alpha}{\alpha}$.

   Since we look near, but not at, $\alpha = 0$, we can justify the cancellation $\lim_{\alpha \to 0} \frac{4\alpha^2 - 2\alpha}{\alpha} = \lim_{\alpha \to 0} 4\alpha - 2 = -2$.

   (c) **(2 points)** $\lim_{x \to -\infty} \frac{x^3 - 3x^2 + 2x - 1}{20x^2}$.

   In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so $\lim_{x \to -\infty} \frac{x^3 - 3x^2 + 2x - 1}{20x^2} = \lim_{x \to -\infty} \frac{x^3}{20x^2} = \lim_{x \to -\infty} \frac{1}{20}x$ which does not exist, since as x decreases without bound, so does $\frac{1}{20}x$.

   (d) **(2 points)** $\lim_{\theta \to \frac{\pi}{3}^-} \frac{\cos\theta}{\theta}$.

   Ins function is continuous except at $\theta = 0$, so the limit is simply a matter of evaluating the expression: $\frac{\cos\frac{\pi}{3}}{\frac{\pi}{3}} = \frac{\frac{1}{2}}{\frac{\pi}{3}} = \frac{3}{2\pi}$.