1. **(8 points)** *The horizontal displacement in meters of a particle at time t, measured in seconds, is given by the formula* $s(t) = t^3 - 8t^2 + 16t$.

   (a) **(2 points)** *What is the particle's velocity after 2 seconds?*

   The velocity function is $s'(t) = 3t^2 - 16t + 16$; thus the velocity after two seconds is $s'(2) = 3(2)^2 - 16(2) + 16 = -4$, which can be interpreted as a speed of -4 meters per second in the leftwards (i.e. negative) direction.

   (b) **(4 points)** *At what times is the particle moving to the right?*

   The question here is equivalent to asking, "when is $s'(t)$ positive?" since rightwards movement corresponds to positive velocity. From above, we have the function $s'(t) = 3t^2 - 16t + 16$, so the question is when $3t^2 - 16t + 16 > 0$; factoring this, we have $s'(t) = (3t - 4)(t - 4)$; so $s'(t)$ is positive when these factors are either both positive (if $t > 4$) or both negative (if $t < \frac{4}{3}$); thus the particule is moving to the right when t is either less than $\frac{4}{3}$ or greater than 4.

   (c) **(2 points)** *What is the particle's acceleration after 5 seconds?*

   Acceleration is given by $s''(t) = 6t - 16$, so after 5 seconds the acceleration is $s''(5) = 6(5) - 16 = 14$, which implies acceleration of 14 meters per second per second *[sic]* to the right.

2. **(8 points)** *A bar of iron is taken from a 1400°F furnace into a 100°F metalworking studio. After 5 minutes it has cooled to 800°F.*

   (a) **(4 points)** *Produce a function $T(t)$ modeling the bar's temperature t minutes after removal from the furnace.*

   We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 100°F, so our temperature model will be $T(t) = 100 + Ce^{-kt}$; it remains only to find C and k to have a final model.

   Since the bar has a temperature of 1400°F immediately upon removal from the furnace, $T(0) = 1400$. Evaluating the left side of this equation, we find that $100 + Ce^0 = 1400$; thus $C = 1300$.

   Since the bar has a temperature of 800°F five minutes later, we know that $T(5) = 800$. Expanding $T(5)$, we find that:

$$\begin{aligned} 100 + 1300e^{-k\cdot 5} &= 800 \\ 1300e^{-5k} &= 700 \\ e^{-5k} &= \frac{7}{13} \\ -5k &= \ln\frac{7}{13} \\ k &= \frac{-\ln\frac{7}{13}}{5} \end{aligned}$$

   Assembling this value of k into our equation, we find that

$$T(t) = 100 + 1300e^{\frac{\ln\frac{7}{13}}{5}t}$$

(b) **(2 points)** *How quickly is the bar's temperature changing immediately upon removal from the furnace?*

The time of the bar's removal from the furnace is definitionally time 0; thus the speed of the bar's cooling at that time is $T'(0)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = 1300\frac{\ln\frac{7}{13}}{5}e^{\frac{\ln\frac{7}{13}}{5}t}$$

so $T'(0) = 1300\frac{\ln\frac{7}{13}}{5}e^0 = 260\ln\frac{7}{13}$. This is approximately -161, signifying that the bar is cooling (dropping in temperature) by 161 degrees per minute.

(c) **(2 points)** *The metal can be worked as long as it is hotter than 1000°F. How soon after the bar is removed from the furnace does it become unworkable?*

Since the metal becomes unworkable when $T(t) = 1000$, we want to find the value of t satisfying that equation:

$$\begin{aligned}
100 + 1300e^{\frac{\ln\frac{7}{13}}{5}t} &= 1000 \\
1300e^{\frac{\ln\frac{7}{13}}{5}t} &= 900 \\
e^{\frac{\ln\frac{7}{13}}{5}t} &= \frac{9}{13} \\
\frac{\ln\frac{7}{13}}{5}t &= \ln\frac{9}{13} \\
t &= \frac{5\ln\frac{9}{13}}{\ln\frac{7}{13}} \approx 3 \text{ minutes}
\end{aligned}$$

3. **(8 points)** *Calculate* $\frac{d}{dx}\left(\sqrt{x}\csc(\ln x)\right)$.

On the most primitive level, the expression being differentiated here is a product, so we apply the product rule:

$$\frac{d}{dx}\left(\sqrt{x}\csc(\ln x)\right) = \left(\frac{d}{dx}\sqrt{x}\right)\csc(\ln x) + \sqrt{x}\frac{d}{dx}\csc(\ln x)$$

Of the two derivatives remaining to be calculated, one is easy: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$; the other requires a chain-rule application. if we let $y = \csc u$ and $u = \ln x$, then since $\frac{dy}{du} = -\csc u \cot u$ and $\frac{du}{dt} = \frac{1}{x}$, we have that

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = (-\csc u \cot u)\frac{1}{x} = \frac{-\csc(\ln x)\cot(\ln x)}{x}$$

so, expanding the known derivatives in the product rule above:

$$\frac{d}{dx}\left(\sqrt{x}\csc(\ln x)\right) = \frac{\csc(\ln x)}{2\sqrt{x}} - \sqrt{x}\frac{\csc(\ln x)\cot(\ln x)}{x}$$

4. **(8 points)** *The* conchoid of de Sluze *is a curve satisfying the equation* $(x-1)(x^2+y^2) = 4x^2$.

(a) **(6 points)** *Find a formula for $\frac{dy}{dx}$ on this curve.*

Taking the derivative of each side, and using rules as necessary:

$$\begin{aligned}
\frac{d}{dx}\left[(x-1)(x^2+y^2)\right] &= \frac{d}{dx}4x^2 \\
\left[\frac{d}{dx}(x-1)\right](x^2+y^2) + (x-1)\frac{d}{dx}(x^2+y^2) &= 8x \\
1(x^2+y^2) + (x-1)(2x + \frac{d}{dx}y^2) &= 8x \\
(x^2+y^2) + (x-1)(2x + \frac{dy}{dx}\frac{d}{dy}y^2) &= 8x \\
(x^2+y^2) + (x-1)(2x + \frac{dy}{dx}2y) &= 8x \\
(x^2+y^2) + 2x^2 - 2x + 2xy\frac{dy}{dx} - 2y\frac{dy}{dx} &= 8x \\
2xy\frac{dy}{dx} - 2y\frac{dy}{dx} &= 10x - 2x^2 - x^2 - y^2 \\
\frac{dy}{dx} &= \frac{10x - 3x^2 - y^2}{2xy - 2y}
\end{aligned}$$

(b) **(2 points)** *Identify conditions on x and y for the tangent lines to the conchoid to be horizontal and vertical (label which is which!).*

The tangent line is horizontal when $\frac{dy}{dx} = 0$, which is the case when its numerator is zero. Thus, the criterion for a tangent line to be horizontal is $10x - 3x^2 - y^2 = 0$.

The tangent line is vertical when $\frac{dy}{dx}$ is undefined due to an infinite asymptote. This will occur when the denominator of $\frac{dy}{dx}$ is zero. Thus, the criterion for a tangent line to be vertical is $2xy - 2y = 0$ (this may, but need not, be simplified to the criterion that either $y = 0$ or $x = 1$.

5. **(8 points)** *Find an equation of the tangent line to the curve $y = 3\sin(\pi x)$ at $(\frac{1}{6}, \frac{3}{2})$.*

Using the chain rule (implicitly or explicitly), $y' = 3\pi\cos(\pi x)$, so when $x = \frac{1}{6}$, $y' = 3\pi\cos(\frac{\pi}{6}) = 3\pi\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}\pi}{2}$. Thus our tangent line has a slope of $\frac{3\sqrt{3}\pi}{2}$ and passes through the point $(\frac{1}{6}, \frac{3}{2})$, so its equation in point-slope form is

$$\left(y - \frac{3}{2}\right) = \frac{3\sqrt{3}\pi}{2}\left(x - \frac{1}{6}\right)$$

6. **(8 points)** *A thief is sneaking eastwards along a wall at two miles per hour. However, 3 miles east and 4 miles south of him, a guard with a telescope is watching.*

(a) **(4 points)** *The guard needs to swivel the telescope to keep it trained on the thief. How quickly should she be turning it?*

Let us name various quantities which will change over time. First, we define time itself measured in hours as t, and name the physical quantities x, the thief's distance to the west of the guard, s, the thief's direct distance from the guard, and θ, the angle between the telescople's lens and due north. Several relationships arise as a result of the problem statement: $x = 3$ at the time in question (although x is not a constant, so it shall not remain 3), since the thief is 3 miles to the west of the guard. $\frac{dx}{dt} = -2$, since the thief's distance to the west of the guard is diminishing by 2 miles per hour. $s^2 = x^2 + 4^2$ by the Pythagorean theorem, and $\tan\theta = \frac{x}{4}$ since x and 4 are the lengths of legs of a right triangle with θ as a corner angle. Here we use the known property of θ, the telescope's angular position to determine $\frac{d\theta}{dt}$, the rate at which the telescope must be turned:

$$\begin{aligned}\tan\theta &= \frac{x}{4}\\ \frac{d}{dt}\tan\theta &= \frac{d}{dt}\frac{x}{4}\\ \frac{d\theta}{dt}\sec^2\theta &= \frac{1}{4}\frac{dx}{dt}\\ \frac{d\theta}{dt} &= \frac{1}{4}\frac{dx}{dt}\cos^2\theta\end{aligned}$$

Since the adjacent side to θ and hypotenuse of the right triangle used in this problem are 4 and s respectively, we know $\cos\theta = \frac{4}{s} = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$. Thus $\frac{d\theta}{dt} = \frac{1}{4}(-2)(\frac{4}{5})^2 = \frac{-32}{100}$, so the telescope must be turned towards due north (clockwise from its current orientation) at $\frac{8}{25}$ radians per hour.

(b) **(4 points)** *How quickly is the distance between the thief and guard changing?*

Here we want $\frac{ds}{dt}$, so we need to use the known property of s above, namely, that $s^2 = x^2 + 4^2$. Attempting to calculate $\frac{ds}{dt}$ therefrom yields:

$$\begin{aligned}\frac{d}{dt}s^2 &= \frac{d}{dt}(x^2+4^2)\\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt}\\ \frac{ds}{dt} &= \frac{2x\frac{dx}{dt}}{2s} = \frac{2\cdot 3(-2)}{2\cdot 5} = \frac{-12}{10}\end{aligned}$$

so the thief is approaching the guard (that is, the distance between them is decreasing) at 1.2 mph.

7. **(8 points)** Differentiate $\frac{(\arcsin u)\sqrt[5]{u}}{\tan u}$ with respect to u.

This is a quotient whose numerator is itself a product (no chain rules were harmed in the making of this derivative), so we need to use the quotient rule and product rule in turn (if preferred, the derivative could be rewritten as a product one of whose factors is a quotient, and the rules could be applied in the opposite order):

$$\begin{aligned}\frac{d}{du}\frac{(\arcsin u)\sqrt[5]{u}}{\tan u} &= \frac{\tan u\frac{d}{du}\left[(\arcsin u)\sqrt[5]{u}\right] - (\arcsin u)\sqrt[5]{u}\frac{d}{du}\tan u}{(\tan u)^2}\\ &= \frac{\tan u\left[\frac{d}{du}(\arcsin u)\sqrt[5]{u} + (\arcsin u)\frac{d}{du}u^{1/5}\right] - (\arcsin u)\sqrt[5]{u}\sec^2 u}{\tan^2 u}\\ &= \frac{\tan u\left[\frac{1}{\sqrt{1-u^2}}\sqrt[5]{u} + (\arcsin u)(\frac{1}{5}u^{-4/5})\right] - (\arcsin u)\sqrt[5]{u}\sec^2 u}{\tan^2 u}\end{aligned}$$