1. **(6 points)** *Evaluate the following limits, or demonstrate that they do not exist:*

   (a) **(2 points)** $\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\theta^2}$.

   We note that, evaluated at $\theta = 0$, the expressions $\cos\theta - 1$ and θ^2 are both zero; thus, our limit involves a $\frac{0}{0}$ indeterminate form, so we are justified in involing L'Hôpital's rule, to find that $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta^2} = \lim_{\theta \to 0} \frac{-\sin\theta}{2\theta}$. Unfortunately, this too is an indeterminate form, as both $-\sin\theta$ and 2θ evaluated at $\theta = 0$ yield zero, so we use L'Hospital's rule again to find that $\lim_{\theta \to 0} \frac{-\sin\theta}{2\theta} = \lim_{\theta \to 0} \frac{-\cos\theta}{2}$, which by direct evaluation can be found to be $-\frac{1}{2}$.

   (b) **(2 points)** $\lim\limits_{x \to 2} \frac{e^x}{x+2}$.

   Evaluated at $x = 2$, the numerator and denominator are e^2 and 4; this is not an indeterminate form, and can be subjected to direct evaluation to yield the result $\frac{e^2}{4}$.

   (c) **(2 points)** $\lim\limits_{x \to \infty} xe^{-x}$.

   Note that $\lim_{x \to \infty} x = +\infty$, and $\lim_{x \to \infty} e^{-x} = 0$. We thus have a $0 \cdot \infty$ indeterminate form occurring in this limit. L'Hospital's rule cannot be applied directly to this form; it must be reformulated as an $\frac{\infty}{\infty}$ or $\frac{0}{0}$ indeterminate form. The easiest way to do so is to rephrase xe^{-x} as $\frac{x}{e^x}$; then note that $\lim_{x \to \infty} x = +\infty$ and $\lim_{x \to \infty} e^x = +\infty$, so $\lim_{x \to \infty} \frac{x}{e^x}$ is an $\frac{\infty}{\infty}$ indeterminate form and may have L'Hospital's rule applied. Thus $\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$.

2. **(8 points)** *You must print a small poster with 200 square inches of printed material and margins of half a inch on the left and right, and a full inch on top and bottom. What dimensions should the poster have to minimize the total area of paper used?*

   Let the width and height of the printed matter be x and y respectively (one could assign variable names to the dimensions of the full sheet instead, but this makes the arithmetic slightly easier). Then the full sheet of paper has width $x + 1$ and height $y + 2$, to accomodate the margins. We are told the area of the printed matter must be 200 square inches, so we have the constraint that $xy = 200$; our goal is to minimize the area of the full sheet, so the expression we seek to minimize is $(x + 1)(y + 2)$. Using our constraint, we express one variable in terms of the other, for instance as $y = \frac{200}{x}$ (it would be equally valid to express x in terms of y). We now substitute this expression for y back into the expression we seek to minimize to make it a function of one variable:

   $$A(x) = (x + 1)\left(\frac{200}{x} + 2\right) = 200 + 2x + \frac{200}{x} + 2$$

   We will minimize this function over the interval $(0, \infty)$, since clearly x must be positive in order to have any text, but there are no other constraints on its value. We find critical points by finding zeroes and nonexistence points of the function's derivative:

   $$A'(x) = 2 - \frac{200}{x^2}$$

   This function is nonexistent at $x = 0$; it is zero when $2 = \frac{200}{x^2}$, which can be rearranged to give $x^2 = 100$, or $x = \pm 10$.

Thus, we have 5 things to take under consideration for optimization purposes: evaluation of the function at $x = 0$, $x = -10$, and $x = 10$,a nd furthermore the limiting behavior of the function as $x \to 0^+$ and $x \to \infty$. Two of these cases may be rejected outright: $x = 0$ and $x = -10$ are outside the interval $(0, \infty)$. we thus consider the others: as $x \to \infty$, $2x$ becomes arbitrarily large, so $A(x)$ will grow without bound; likewise, as $x \to 0^+$, $\frac{200}{x}$ will grow without bound. Thus, neither of these is a plausible minimizing behavior. Our only remaining choice for minimum is $x = 10$, which yields $y = \frac{200}{10} = 20$, so our poster as a whole would have dimensions $(10 + 1) \times (20 + 2) = 11 \times 22$. The area of this sheet would be 242 square inches, which is not an unreasonable size for a sheet with 200 square inches printed and generous margins.

3. **(6 points)** *Answer the following questions about the function* $f(x) = (2x - 5)e^x$.

(a) **(2 points)** *For which values of x is it increasing? For which is it decreasing? Label which is which.*

Increase and decrease are dictated by the sign of $f'(x)$, so we start by determining $f'(x)$ with the product rule:

$$f'(x) = (2)e^x + (2x - 5)e^x = (2x - 3)e^x$$

With $f'(x)$ expressed as a product, it is easy to determine its sign by determining the signs of its constituent parts. e^x is positive everywhere; $(2x - 3)$ is positive when $x > \frac{3}{2}$ and negative when $x < \frac{3}{2}$ (and zero when $x = \frac{3}{2}$). Thus, $f(x)$ is increasing when $x > \frac{3}{2}$, and decreasing when $x < \frac{3}{2}$.

(b) **(2 points)** *Find all local extrema of the function and identify each as a local maximum or local minimum.*

The function $f'(x)$ identified above exists everywhere, and was shown above to be zero at $x = \frac{3}{2}$. We thus have a single critical point at $x = \frac{3}{2}$. Since the function decreases up to this critical point, and increases after it, we know it to be a local minimum of the function.

(c) **(2 points)** *For what values of x is the function concave up, and when is it concave down? Label which is which. Identify all points of inflection of the function.*

Concavity is dependent on the sign of $f''(x)$, so we must find $f''(x)$, again using the product rule:

$$f''(x) = \frac{d}{dx}\left[(2x - 3)e^x\right] = 2e^x + (2x - 3)e^x = (2x - 1)e^x$$

With $f''(x)$ expressed as a product, it is easy to determine its sign by determining the signs of its constituent parts. e^x is positive everywhere; $(2x - 1)$ is positive when $x > \frac{1}{2}$ and negative when $x < \frac{1}{2}$ (and zero when $x = \frac{1}{2}$). Thus, $f(x)$ is concave up when $x > \frac{1}{2}$, and concave down when $x < \frac{1}{2}$; in addition, since it transitions between the two concavities at $x = \frac{1}{2}$, we have a point of inflection at $x = \frac{1}{2}$.