1. **(20 points)** *Evaluate the following integrals:*

   (a) **(10 points)** $\int \frac{dt}{t^2+2t+17}$.
   This is an irreducible quadratic; its denominator can thus be rephrased via completion of the square as a sum of two squares, which under appropriate division becomes an expression of the form u^2+1, which can be integrated using an implicit linear substitution:

$$\begin{aligned}\int \frac{dt}{t^2+2t+17} &= \int \frac{dt}{t^2+2t+1+16} \\ &= \int \frac{dt}{(t+1)^2+4^2} \\ &= \int \frac{\frac{1}{4^2}dt}{\left(\frac{t+1}{4}\right)^2+1^2} \\ &= \frac{1}{16}\int \frac{dt}{\left(\frac{t+1}{4}\right)^2+1^2} \\ &= \frac{1}{16}\cdot 4\arctan\frac{t+1}{4}+C = \frac{1}{4}\arctan\frac{t+1}{4}+C\end{aligned}$$

   (b) **(10 points)** $\int \frac{x+6}{x(x-3)(x+2)}dx$. For this factorization into distinct linear terms, the appropriate decomposition is

$$\frac{x+6}{x(x-3)(x+2)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-3}$$

   which, on multiplying by the common denominator, yields

$$\begin{aligned}x+6 &= A(x-3)(x+2)+Bx(x-3)+Cx(x+2) \\ x+6 &= A(x^2-x-6)+B(x^2-3x)+C(x^2+2x) \\ 0x^2+1x+6 &= (A+B+C)t^2+(-A-3B+2C)t-6A\end{aligned}$$

   Comparing quadratic, linear, and constant terms on the left and right side of the above equation yields the system of equations

$$\begin{cases} A+B+C=0 \\ -A-3B+2C=1 \\ -6A=6 \end{cases}$$

   The last equation gives us $A=-1$ immediately; combined with the first two, we see that $B+C=1$ and $-3B+2C=0$, which can be reduced to $B=\frac{2}{5}$ and $C=\frac{3}{5}$. Thus

$$\frac{x+6}{x(x-3)(x+2)} = -\frac{1}{x}+\frac{\frac{2}{5}}{x+2}+\frac{\frac{3}{5}}{x-3}$$

   so that

$$\int \frac{x+6}{x(x-3)(x+2)}dx = -\int \frac{1}{x}+\frac{\frac{2}{5}}{x+2}+\frac{\frac{3}{5}}{x-3}dx = -\ln|x|+\frac{2}{5}\ln|x+2|+\frac{3}{5}\ln|x-3|+C$$

2. **(20 points)** *Evaluate the following integrals:*

(a) **(10 points)** $\int t^3\sqrt{4-t^2}dt$.
The term $\sqrt{4-t^2}$ suggests a trigonometric substitution, and in particular a sine substitution. If we construct a right triangle with marked angle θ, hypotenuse length 2, opposite side length t, and adjacent side length $\sqrt{4-t^2}$, we find that $t = 2\sin\theta$ and $\sqrt{4-t^2} = 2\cos\theta$. Furthermore, for the substitution, we will need the differential $dt = 2\cos\theta d\theta$. Then the above integral becomes

$$\int (2\sin\theta)^3(2\cos\theta)(2\cos\theta)d\theta = \int 2^5\sin^3\theta\cos^2\theta d\theta$$

Since the above integral has an odd number of multiplicative $sin\theta$ terms, judicious use of the identity $\sin^2\theta = (1-\cos^2\theta)$ will give an expression solely in terms of $\cos\theta$ save for a single multiplicative term $\sin\theta$:

$$\int 2^5\sin^3\theta\cos^2\theta d\theta = \int 2^5\sin\theta(1-\cos^2\theta)\cos^2\theta d\theta = \int 2^5(\cos^2\theta-\cos^4\theta)\sin\theta d\theta$$

This form is now ripe for the substitution $u=\cos\theta$, $du = -\sin\theta d\theta$, which will serve to simplify this integral:

$$\begin{aligned}\int 2^5(\cos^2\theta-\cos^4\theta)\sin\theta d\theta &= \int 2^5(u^2-u^4)(-du)\\ &= 2^5\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C\\ &= \frac{(2\cos\theta)^5}{5}-\frac{4(2\cos\theta)^3}{3}+C\\ &= \frac{\sqrt{4-t^2}^5}{5}-\frac{4\sqrt{4-t^2}^3}{3}+C\end{aligned}$$

(b) **(10 points)** $\int \frac{4y}{\sqrt{y^2+9}}dy$.
The term $\sqrt{y^2+9}$ suggests a trigonometric substitution, and in particular a tangent substitution. If we construct a right triangle with marked angle θ, hypotenuse length $\sqrt{y^2+9}$, adjacent side length 3, and opposite side length y, we find that $y = 3\tan\theta$ and $\sqrt{t^2+9} = 3\sec\theta$. Furthermore, for the substitution, we will need the differential $dt = 3\sec^2\theta d\theta$. Then the above integral becomes

$$\int \frac{4(3\tan\theta)}{(3\sec\theta)}(3\sec^2\theta d\theta) = \int 12\sec\theta\tan\theta d\theta$$

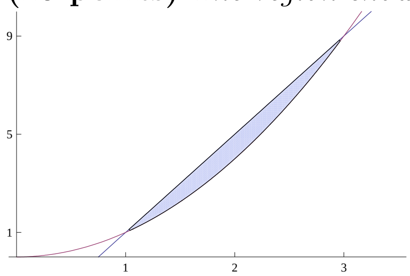
which is actually pretty easily solved:

$$\int 12\sec\theta d\theta = 12\sec\theta + C = 4\sqrt{y^2+9}+C$$

Note that this problem could also be solved, and arguably much more simply, by using a standard substitution, since $\sqrt{y^2+9}$, in addition to being a trigonometric form, is a composition of a square root with the expression y^2+9, and the derivative of y^2+9 appears in a modified form as a factor of the integrand, suggesting the substitution $u = y^2+9$, which gives $du = 2ydy$, allowing us to rephrase the integral like so:

$$\int \frac{4y}{\sqrt{y^2+9}}dy = \int \frac{2du}{\sqrt{u}} = 2\frac{u^{1/2}}{1/2}+C = 4\sqrt{y^2+9}+C$$

3. **(10 points)** *The region shown below is the area between the curves $y = 4x - 3$ and $y = x^2$.*



(a) **(5 points)** *Find the area of this region.*

The region is bounded on the left by $x = 1$ and on the right by $x = 3$. The upper curve is $y = 4x - 3$ and the lower curve is $y = x^2$. Thus, the integral to calculate the area is

$$\int_1^3 (4x-3) - x^2 dx = 2x^2 - 3x - \frac{x^3}{3}\Big]_1^3 = (18 - 9 - 9) - (2 - 3 - \frac{1}{3}) = \frac{4}{3}$$

(b) **(5 points)** *Find the volume of the solid produced by rotating this region around the x-axis.*

The bounds of the region are as given above; in this case, spinning the region around the x-axis, the vertical cross-sections whose height formed the integrands above trace out washers of outer radius $4x - 3$ and inner radius x^2. Thus the integral to give the volume of the solid is

$$\begin{aligned}\int_1^3 \pi\left(4x-3\right)^2 - \pi\left(x^2\right)^2 dx &= \pi \int_1^3 16x^2 - 24x + 9 - x^4 dx \\ &= \pi\left[\frac{16x^3}{3} - 12x^2 + 9x - \frac{x^5}{5}\right]_1^3 \\ &= \pi\left(\frac{16 \cdot 27}{3} - 12 \cdot 9 + 9 \cdot 3 - \frac{3^5}{5}\right) - \left(\frac{16}{3} - 12 + 9 - \frac{1}{5}\right) = \frac{184\pi}{15}\end{aligned}$$

4. **(20 points)** *Evaluate the following integrals:*

(a) **(10 points)** $\int (x^2 - 2x)e^x dx$.

This is a product of a polynomial with the integrable function e^x, so we will integrate by parts, making use of the decomposition $u = x^2 - 2x$, $dv = e^x dx$, which gives $du = (2x - 2)dx$ and $v = e^x$. Integration by parts thus gives:

$$\int (x^2 - 2x)e^x dx = (x^2 - 2x)e^x - \int (e^x)(2x - 2)dx$$

The new integral we have at the end of this line is likewise a product of a polynomial and an integrable function, so again we use integration by parts, with the decomposition $u = 2x - 2$, $dv = e^x dx$, which gives $du = 2dx$ and $v = e^x$. Continuing the integration by parts:

$$\begin{aligned}\int (x^2-2x)e^x dx &= (x^2-2x)e^x - \int (2x-2)e^x dx \\ &= (x^2-2x)e^x - \left((2x-2)e^x - \int e^x(2dx)\right) \\ &= (x^2-2x)e^x - \left((2x-2)e^x - 2\int e^x dx\right) \\ &= (x^2-2x)e^x - ((2x-2)e^x - 2e^x) + C \\ &= (x^2-4x+4)e^x + C\end{aligned}$$

(b) **(10 points)** $\int 4x \sin 5x dx$.

This is a product of a polynomial with the integrable function $\sin 5x$, so we will integrate by parts, making use of the decomposition $u = 4x$, $dv = \sin 5x dx$, which gives $du = 4dx$ and $v = \frac{-1}{5}\cos 5x$. Integration by parts thus gives:

$$\begin{aligned}\int 4x \sin 5x dx &= 4x \cdot \frac{-1}{5}\cos 5x - \int \frac{-1}{5}\cos 5x(4dx) \\ &= \frac{-4}{5}x\cos 5x + \frac{4}{5}\int \cos 5x dx \\ &= \frac{-4}{5}x\cos 5x + \frac{4}{25}\sin 5x + C\end{aligned}$$

5. **(15 points)** *Evaluate the following integrals:*

(a) **(5 points)** $\int \sin x e^{\cos x} dx$.

There is a composition in this integrand: $e^{\cos x}$ naturally decomposes as a composition of the exponential function and the expression $\cos x$; furthermore, the derivative of $\cos x$ appears in slightly modified form as a factor of the integrand, so the substitution $u = \cos x$ is strongly suggested. Letting $u = \cos x$, it follows that $du = -\sin x dx$. Converting the integral above using this substitution thus yields:

$$\int \sin x e^{\cos x} dt = \int e^u(-du) = -e^u + C = -e^{\cos x} + C$$

(b) **(5 points)** $\int \tan^6\theta \sec^2\theta d\theta$.

There is a composition in this integrand: $\tan^6\theta$ naturally decomposes as a composition of the sixth power and the expression $\tan\theta$; furthermore, the derivative of $\tan\theta$ appears as a factor of the integrand, so the substitution $u = \tan\theta$ is strongly suggested. Letting $u = \tan\theta$, it follows that $du = \sec^2\theta d\theta$. Converting the integral above using this substitution thus yields:

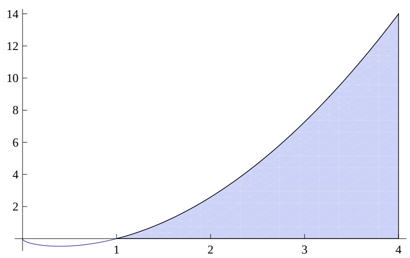
$$\int \tan^6\theta\sec^2\theta d\theta = \int u^6 du = \frac{u^7}{7} + C = \frac{\tan^7\theta}{7} + C$$

(c) **(5 points)** $\int_1^2 x^3 e^{(x^4)} dx$.

There is a composition in this integrand: $e^{(x^4)}$ naturally decomposes as a composition of the exponential function and the expression x^4; furthermore, the derivative of x^4 appears in slightly modified form as a factor of the integrand, so the substitution $u = x^4$ is strongly suggested. Letting $u = x^4$, it follows that $du = 3x^3dx$, or more ot our purpose, $\frac{du}{3} = x^3dx$. Converting the integral above using this substitution thus yields:

$$\int_1^2 x^3e^{x^4}dt = \int_{x=1}^{x=2} e^u\frac{du}{3} = \frac{e^u}{3}\Bigg]_{x=1}^{x=2} = \frac{e^{x^4}}{3}\Bigg]_1^2 = \frac{e^{16}-e^1}{3}$$

6. **(15 points)** *The region shown below is the area under the curve $y = x^2 - \sqrt{x}$ from $x = 1$ to $x = 4$.*



(a) **(5 points)** *Construct, but do not evaluate, an integral representing the volume of the solid produced by rotating this region around the x-axis.*

Working from the left bound ($x = 1$) to the right bound ($x = 4$) in the horizontal direction, the cross-sections of this solid will be discs of radius $x^2 - \sqrt{x}$, so the integral setup is

$$\int_1^4 \pi(x^2 - \sqrt{x})^2dx$$

(b) **(5 points)** *Construct, but do not evaluate, an integral representing the volume of the solid produced by rotating this figure around the y-axis.*

This cannot be set up as a disc problem: doing so would require inverting the function $f(x) = x^2 + \sqrt{x}$, a matter of some difficulty. Thus we still need to integrate with respect to x, and thus the vertical cross-sections at particular x-values revolve out into cylindrical shells of radius x and height $x^2 - \sqrt{x}$. Thus, the volume is represented by the integral

$$\int_1^4 2\pi x(x^2 - \sqrt{x})dx$$

(c) **(5 points)** *Calculate the average value of the function $f(x) = x^2 - \sqrt{x}$ on the interval $[1, 4]$.*

The average value is

$$\frac{\int_1^4 x^2 - \sqrt{x}}{4-1} = \frac{\left[\frac{x^3}{3} - \frac{x^{3/2}}{3/2}\right]_1^4}{3} = \frac{\left(\frac{64}{3} - \frac{8}{3/2}\right) - \left(\frac{1}{3} - \frac{1}{3/2}\right)}{3} = \frac{(64-16)-(1-2)}{9} = \frac{49}{9}$$