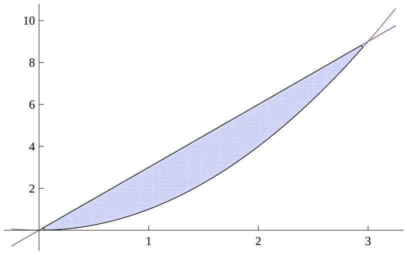
1. **(15 points)** *The region shown below is the area between the curves $y = 3x$ and $y = x^2$. Find the center of mass of this region.*



We must calculate the area, x-moment, and y-moment to find the center of mass.

The region is bounded on the left by $x = 0$ and on the right by $x = 3$. The upper curve is $y = 3x$ and the lower curve is $y = x^2$. Thus, the integral to calculate the area is

$$A = \int_0^3 3x - x^2 dx = \frac{3x^2}{2} - \frac{x^3}{3}\bigg]_0^3 = \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0) = \frac{9}{2}$$

To calculate the x=moment, we multiply the integrand above by x to get

$$M_x = \int_0^3 3x^2 - x^3 dx = x^3 - \frac{x^4}{4}\bigg]_0^3 = \left(27 - \frac{81}{4}\right) - (0 - 0) = \frac{27}{4}$$

To calculate the y-moment, however, we need to use one half the difference of the squares of the upper and lower functions:

$$M_y = \int_0^3 \frac{1}{2}(3x)^2 - \frac{1}{2}\left(x^2\right)^2 dx = \frac{1}{2}\int_0^3 9x^2 - x^4 dx = \frac{1}{2}\left[3x^3 - \frac{x^5}{5}\right]_0^3 = \frac{1}{2}\left(81 - \frac{243}{5}\right) - (0 - 0) = \frac{81}{5}$$

Thus, the center of mass of the above region is $\left(\frac{M_x}{A}, \frac{M_y}{A}\right) = \left(\frac{3}{2}, \frac{18}{5}\right)$.

2. **(10 points)** *Find the general solution to the differential equation $\frac{dy}{dx} - 2xy = 4e^{x^2-2x}$.*

This is a linear differential equation, with coefficient $-2x$ on the y-term. Thus, we calculate the integrating factor:

$$\rho = e^{\int -2x dx} = e^{-x^2}$$

Multiplying the whole equation by this integrating factor, we see (or hope, at least) that the left side folds into a single derivative, which we can then integrate:

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 4e^{x^2-2x}e^{-x^2}$$
$$\frac{d}{dx}\left(e^{-x^2}y\right) = 4e^{-2x}$$
$$e^{-x^2}y = \int 4e^{-2x}dx$$
$$e^{-x^2}y = \frac{4}{-2}e^{-2x} + C$$
$$y = -2e^{x^2-2x} + Ce^{x^2}$$

3. **(15 points)** *Evaluate the following integrals, or if they cannot be evaluated, demonstrate why not.*

   (a) **(7 points)** $\int_{-2}^{4} \frac{1}{x} dx$.

   The function $\frac{1}{x}$ has a discontinuity at $x = 0$, so the above integral is improper and must be rephrased as a sum of limites of definite integrals:

$$\begin{aligned}\int_{-2}^{4} \frac{1}{x} dx &= \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{x} dx + \lim_{a \to 0^+} \int_{a}^{4} \frac{1}{x} dx \\ &= \lim_{b \to 0^-} \ln|x|\Big]_{-2}^{b} + \lim_{a \to 0^+} \ln|x|\Big]_{a}^{4} \\ &= \lim_{b \to 0^-} \ln|b| - \ln 2 + \lim_{a \to 0^+} \ln 4 - \ln|a| \\ &= \ln 4 - \ln 2 + \lim_{b \to 0^-} \ln|b| - \lim_{a \to 0^+} \ln|a|\end{aligned}$$

   Since zero has no natural logarithm, or and there is not even a limit to the natural logarithm as its argument approaches zero, neither of the limits shown above exist, so the integral is divergent.

   (b) **(8 points)** $\int_{5}^{\infty} \frac{1}{\sqrt[3]{x-4}} dx$.

   Using limits to rephrase this improper integral:

$$\begin{aligned}\int_{5}^{\infty} \frac{1}{\sqrt[3]{x-4}} dx &= \lim_{b \to \infty} \int_{5}^{b} (x-4)^{-1/3} dx \\ &= \lim_{b \to \infty} \frac{(x-4)^{2/3}}{2/3}\Big]_{5}^{b} \\ &= \lim_{b \to \infty} \frac{(b-4)^{2/3}}{2/3} - \frac{(5-4)^{2/3}}{2/3}\end{aligned}$$

   However, as b grows without bound, so does $(b-4)^{2/3}$, so this integral is divergent.

4. **(15 points)** *Consider the function* $f(x) = \begin{cases} 0 \text{ for } x < 4 \\ \frac{k}{x^{5/2}} \text{ for } x \geq 4 \end{cases}$ *with k a constant.*

   (a) **(6 points)** *Find a value of k such that $f(x)$ is a probability distribution function.*

   A cursory inspection reveals that this function is non-negative throughout if k is positive: 0 is non-negative everywhere, and $\frac{1}{x^{5/2}}$ is non-negative in its entire domain. The critical property to demonstrate that this function is a probability distribution function is simply that $\int_{-\infty}^{\infty} f(x) dx = 1$. We can simplify this somewhat by ignoring the region on which

$f(x)$ is zero, so that $\int_{-\infty}^{\infty} f(x) = \int_4^{\infty} f(x)dx$. We evaluate this as such:

$$\begin{aligned}\int_4^{\infty} f(x)dx &= \int_4^{\infty} \frac{k}{x^{5/2}}dx \\ &= \lim_{b\to\infty} \int_4^{b} \frac{k}{x^{5/2}}dx \\ &= \lim_{b\to\infty} \frac{-2k}{3x^{3/2}}\Big]_4^b \\ &= \lim_{b\to\infty} \frac{2k}{3b^{3/2}} + \frac{2k}{3\cdot 4^{3/2}} \\ &= 0 + \frac{2k}{24}\end{aligned}$$

so for $\frac{2k}{24}$ to equal 1, we would choose $k = 12$.

(b) **(6 points)** *For a random variable X described by the above probability distribution function, find the average value of X.*

The expected value (or average value) of a probability distribution function $f(x)$ is $\int_{-\infty}^{\infty} xf(x)dx$. We may ignore locations where the integrand is zero, so this can be simplified to $\int_4^{\infty} xf(x)dx$:

$$\begin{aligned}\int_4^{\infty} xf(x)dx &= \int_4^{\infty} x\frac{12}{x^{5/2}}dx \\ &= \lim_{b\to\infty} \int_4^{b} \frac{12}{x^{3/2}}dx \\ &= \lim_{b\to\infty} \frac{-24}{x^{1/2}}\Big]_4^b \\ &= \lim_{b\to\infty} \frac{-24}{b^{1/2}} + \frac{24}{4^{1/2}} \\ &= 0 + \frac{24}{2} = 12\end{aligned}$$

(c) **(3 points)** *For a random variable X described by the above probability distribution function, find $P(X \leq 9)$.*

This probability will be simply $\int_4^9 f(x)dx$. It is techincally $\int_{-\infty}^9 f(x)dx$, but we can ignore the region over which the function is zero.

$$\int_4^9 f(x)dx = \int_4^9 \frac{12}{x^{5/2}}dx = \frac{-8}{x^{3/2}}\Big]_4^9 = \frac{-8}{9^{3/2}} + \frac{8}{4^{3/2}} = \frac{-8}{27} + 1 = \frac{19}{27}$$

5. **(10 points)** *Consider the curve $y = e^x + 4$ between the points $(0,4)$ and $(2, 4+e^2)$.*

(a) **(4 points)** *Construct, but do not evaluate, an integral representing the length of this curve.*

The arclength expression $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ evaluates in this case to $\sqrt{1+(e^x)^2}$, so the arclength is

$$\int_0^4 \sqrt{1+e^{2x}}dx$$

(b) **(3 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the vertical line $x = -3$.*

Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1+e^{2x}}dx$, being spun around circles of radius $x+3$, since the horizontal distance between the line $x = -3$ and the point $(x, e^x + 4)$ is $x + 3$; thus the differential area traced out is $2\pi(x+3)\sqrt{1+e^{2x}}dx$, so the integral to compute the total surface area is

$$\int_0^4 2\pi(x+3)\sqrt{1+e^{2x}}dx$$

(c) **(3 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the x-axis.*

Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1+e^{2x}}dx$, being spun around circles of radius e^x+4, since the vertical distance between the line $y = 0$ and the point (x, e^x+4) is simply e^x+4; thus the differential area traced out is $2\pi(e^x+4)\sqrt{1+e^{2x}}dx$, so the integral to compute the total surface area is

$$\int_0^4 2\pi(e^x+4)\sqrt{1+e^{2x}}dx$$

6. **(15 points)** *Perform the approximations shown below.*

(a) **(5 points)** *Using Simpson's rule with $n = 6$, approximate $\int_1^4 \frac{1}{x}dx$. You need not arithmetically simplify your result.*

Since the interval has length 3 and $n = 6$, the choice of Δx is $\frac{3}{6} = \frac{1}{2}$. Thus, we will need to sample the integrand at the seven points $x = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$. Assembling these into Simpson's rule, we would find that the approximation is:

$$\frac{1/2}{3}\left(\frac{1}{1}+\frac{4}{3/2}+\frac{2}{2}+\frac{4}{5/2}+\frac{2}{3}+\frac{4}{7/2}+\frac{1}{4}\right)$$

If this were simplified, we would have $\frac{3497}{2520}$, which is not a particularly bad rational estimate for the actual integral, which is $\ln 4$.

(b) **(10 points)** *Consider the differential equation $\frac{dy}{dx} = 4xy$ subject to the initial condition that when $x = 1$, $y = -5$. Using Euler's method with a step size of 0.5, approximate the value of y when $x = 2$.*

We have a differential equation whose slope (i.e. $\frac{dy}{dx}$) at each point is described by the function $m(x, y) = 4xy$. We will be using Euler's method on this with $\Delta x = 0.5$ and initial point of $(x_0, y_0) = (1, -5)$. From this, we will calculate new positions x_1 and y_1.

$$x_1 = x_0 + \Delta x = 1 + 0.5 = 1.5$$

$$y_1 = y_0 + \Delta x m(x_0, y_0) = -5 + 0.5 \cdot 4 \cdot 1(-5) = -15$$

so the second point in our estimation of this curve is $(1.5, -15)$. We repeat Euler's method at this new point to find x_2 and y_2:

$$x_2 = x_1 + \Delta x = 1.5 + 0.5 = 2$$

$$y_2 = y_1 + \Delta x m(x_1, y_1) = -15 + 0.5 \cdot 4 \cdot (1.5)(-15) = -60$$

so the third point in our estimation of this curve is $(2, -60)$; thus, when $x = 2$, we estimate y to be -60.

Note that this is a separable differential equation: were it to be solved, one would get the result $y = -5e^{2x^2-2}$, so the correct value of y when $x = 2$ is in fact $-5e^6$, which is far less than -60.

7. **(20 points)** *Answer the following questions about the differential equation* $\frac{dy}{dx} = \frac{e^x}{y}$.

   (a) **(5 points)** *Demonstrate without explicitly solving the differential equation that* $y = \sqrt{2}e^{0.5x}$ *is a solution.*

   Evaluating the left and right sides of the differential equation, and substituting in this solution, we get the two expressions:

$$\frac{d}{dx}\sqrt{2}e^{0.5x} = \sqrt{2}(0.5)e^{0.5x}$$

$$\frac{e^x}{\sqrt{2}e^{0.5x}} = \frac{e^{0.5x}}{\sqrt{2}}$$

   which are identical, given that $\sqrt{2}(0.5) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.

   (b) **(10 points)** *Find the general solution of the differential equation.*

   Using separable methods, we rearrange this integral into

$$\begin{aligned} y\frac{dy}{dx} &= e^x \\ \int y dy &= \int e^x dx \\ \frac{1}{2}y^2 &= e^x + C \\ y^2 &= 2e^x + C \\ y &= \sqrt{2e^x + C} \end{aligned}$$

   Note that the specific solution given in part (a) is this general solution with $C = 0$.

   (c) **(5 points)** *Using your general solution, find a solution to the differential equation subject to the initial condition that* $y = 4e$ *when* $x = 2$.

   Plugging $y = 4e$ and $x = 2$ into the solution found above: $4e = \sqrt{2e^2 + C}$, leading to $16e^2 = 2e^2 + C$, so $C = 14e^2$, leading to the specific solution $y = \sqrt{2e^t + 14e^2}$.