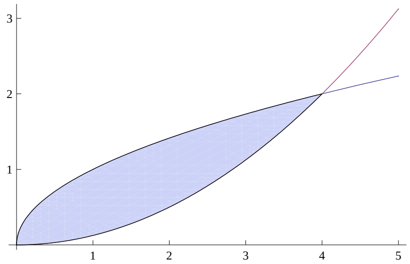
Note: in accordance with requests to limit the scope of the exam, this practice exam does not, nor will the actual exam, cover sections 7.7 and 7.8. You will, however, be expected to know these sections for Quiz #3 and Exam #2.

This test is closed-book and closed-notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified.

The problems are in no particular order, and it is suggested that you look at all of them before beginning to answer any.

1. **(10 points)** *The region shown below is the area between the curves* $y = \sqrt{x}$ *and* $y = \frac{1}{8}x^2$.



(a) **(5 points)** *Find the area of this region.*

The region is bounded on the left by $x = 0$ and on the right by $x = 4$. The upper curve is $y = \sqrt{x}$ and the lower curve is $y = \frac{1}{8}x^2$. Thus, the integral to calculate the area is

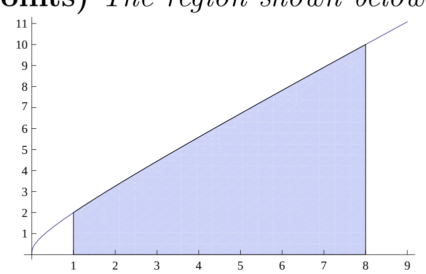
$$\int_0^4 \sqrt{x} - \frac{1}{8}x^2 dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{24}\Bigg]_0^4 = \left(\frac{8}{3/2} - \frac{64}{24}\right) - (0-0) = \frac{8}{3}$$

(b) **(5 points)** *Find the volume of the solid produced by rotating this region around the* x*-axis.*

The bounds of the region are as given above; in this case, spinning the region around the x-axis, the vertical cross-sections whose height formed the integrands above trace out washers, or outer radius $\sqrt{x}$ and inner radius $\frac{1}{8}x^2$. Thus the integral to give the volume of the solid is

$$\int_0^4 \pi\left(\sqrt{x}\right)^2 - \pi\left(\frac{1}{8}x^2\right)^2 dx = \pi\int_0^4 x - \frac{x^4}{64} dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{320}\right]_0^4 = \left(\frac{16}{2} - \frac{4^5}{5 \cdot 64}\right) - (0-0) = \frac{24}{5}$$

2. **(15 points)** *The region shown below is the area under the curve* $y = x + \sqrt[3]{x}$ *from* $x = 1$ *to* $x = 8$.



(a) **(5 points)** *Construct, but do not evaluate, an integral representing the volume of the solid produced by rotating this figure around the* x*-axis.*

Working from the left bound ($x = 1$) to the right bound ($x = 8$) in the horizontal direction, the cross-sections of this solid will be discs of radius $x + \sqrt[3]{x}$, so the integral setup is

$$\int_0^8 \pi(x + \sqrt[3]{x})^2 dx$$

(b) **(5 points)** *Construct, but do not evaluate, an integral representing the volume of the solid produced by rotating this figure around the line $x = -2$.*

This cannot be set up as a disc problem: doing so would require inverting the function $f(x) = x + \sqrt[3]{x}$, a matter of some difficulty. Thus we still need to integrate with respect to x, and thus the vertical cross-sections at particular x-values revlove out into cylindrical shells of radius $x - (-2) = x + 2$ and height $x + \sqrt[3]{x}$. Thus, the volume is represented by the integral

$$\int_0^8 2\pi(x+2)(x+\sqrt[3]{x})dx$$

(c) **(5 points)** *Calculate the average value of the function $f(x) = x + \sqrt[3]{x}$ on the interval $[1, 8]$.*

The average value is

$$\frac{\int_1^8 x + \sqrt[3]{x}}{8-1} = \frac{\left[\frac{x^2}{2} + \frac{x^{4/3}}{4/3}\right]_1^8}{7} = \frac{\left(\frac{64}{2} + \frac{16}{4/3}\right) - \left(\frac{1}{2} + \frac{1}{4/3}\right)}{7} = \frac{(32+12) - (\frac{1}{2} + \frac{3}{4})}{7} = \frac{171}{28}$$

3. **(15 points)** *Evaluate the following integrals:*

(a) **(5 points)** $\int x^2 \sec^2(x^3)dx$.

There is a composition in this integrand: $\sec^2(x^3)$ naturally decomposes as a composition of the known-integrable square of a secant, and the expression x^3; furthermore, the derivative of x^3 appears in slightly modified form as a factor of the integrand, so the substitution $u = x^3$ is strongly suggested. Letting $u = x^3$, it follows that $du = 3x^2dx$, which can be expressed more to our purpose as $\frac{du}{3} = x^2dx$. Converting the integral above using this substitution thus yields:

$$\int x^2 \sec^2(x^3)dx = \int \sec^2 u \frac{du}{3} = \frac{\tan u}{3} + C = \frac{\tan(x^3)}{3} + C$$

(b) **(5 points)** $\int_0^{\pi/2} \sin\theta \cos^3\theta d\theta$.

There is a composition in this integrand: $\cos^3\theta$ naturally decomposes as a composition of a cubing and the expression $\cos\theta$; furthermore, the derivative of $\cos\theta$ appears in slightly modified form as a factor of the integrand, so the substitution $u = x^3$ is strongly suggested. Letting $u = \cos\theta$, it follows that $du = -\sin\theta d\theta$, which can be expressed more to our purpose as $-du = \sin\theta d\theta$. Converting the integral above using this substitution thus yields:

$$\int_0^{\pi/2} \sin\theta\cos^3\theta d\theta = \int_{\theta=0}^{\theta=\pi/2} u^3(-du) = \left.\frac{-u^4}{4}\right]_{\theta=0}^{\theta=\pi/2} = \left.\frac{-\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{-\cos^4\frac{\pi}{2} + \cos^4 0}{4} = \frac{1}{4}$$

(c) **(5 points)** $\int (4t-10)e^{(t^2-5t+3)}dt$.

There is a composition in this integrand: $e^{(t^2-5t+3)}$ naturally decomposes as a composition of the exponential function and the expression $t^2 - 5t + 3$; furthermore, the derivative of $t^2 - 5t + 3$ appears in slightly modified form as a factor of the integrand, so the substitution $u = t^2 - 5t + 3$ is strongly suggested. Letting $u = t^2 - 5t + 3$, it follows that $du = 2t - 5dt$. Converting the integral above using this substitution thus yields:

$$\int (4t-10)e^{(t^2-5t+3)}dt = \int 2e^u du = 2e^u + C = 2e^{t^2-5t+3} + C$$

4. **(20 points)** *Evaluate the following integrals:*

   (a) **(10 points)** $\int (3x+1)e^{-x}dx$.

   This is a product of a polynomial with the integrable function e^{-x}, so we will integrate by parts, making use of the decomposition $u = 3x+1$, $dv = e^{-x}dx$, which gives $du = 3dx$ and $v = -e^{-x}$. Integration by parts thus gives:

$$\begin{aligned}\int (3x+1)e^{-x}dx &= (3x+1)(-e^{-x}) - \int (-e^{-x})(3dx)\\ &= (-3x-1)e^{-x} + 3\int e^{-x}dx\\ &= (-3x-1)e^{-x} - 3e^{-x} + C = (-3x-4)e^{-x} + C\end{aligned}$$

   (b) **(10 points)** $\int (x^2-1)\sin x dx$.

   This is a product of a polynomial with the integrable function $\sin x$, so we will integrate by parts, making use of the decomposition $u = x^2-1$, $dv = \sin x dx$, which gives $du = 2xdx$ and $v = -\cos x$. Integration by parts thus gives:

$$\int (x^2-1)\sin x dx = (x^2-1)(-\cos x) - \int (-\cos x)(2xdx) = (1-x^2)\cos x + 2\int x\cos x dx$$

   The new integral we have at the end of this line is likewise a product of a polynomial and an integrable function, so again we use integration by parts, with the decomposition $u = x$, $dv = \cos x dx$, which gives $du = dx$ and $v = \sin x$. Continuing the integration by parts:

$$\begin{aligned}\int (x^2-1)\sin x dx &= (1-x^2)\cos x + 2\int x\cos x dx\\ &= (1-x^2)\cos x + 2\left(x\sin x - \int \sin x(dx)\right)\\ &= (1-x^2)\cos x + 2\left(x\sin x + \cos x + C\right)\\ &= (3-x^2)\cos x + 2x\sin x + C\end{aligned}$$

5. **(20 points)** *Evaluate the following integrals:*

   (a) **(10 points)** $\int t\sqrt{t^2-4}dt$.

   The term $\sqrt{t^2-4}$ suggests a trigonometric substitution, and in particular a secant substitution. If we construct a right triangle with marked angle θ, hypotenuse length t, adjacent side length 2, and opposite side length $\sqrt{t^2-4}$, we find that $t = 2\sec\theta$ and $\sqrt{t^2-4} = 2\tan\theta$. Furthermore, for the substitution, we will need the differential $dt = 2\sec\theta\tan\theta d\theta$. Then the above integral becomes

$$\int (2\sec\theta)(2\tan\theta)(2\sec\theta\tan\theta d\theta) = \int 8\tan^2\theta\sec^2\theta d\theta$$

   Since the above integral has a $\sec^2\theta$ term and is otherwise in terms of $\tan\theta$, the substitution $u = \tan\theta$, $du = \sec^2\theta d\theta$ will serve to simplify this integral:

$$\int 8\tan^2\theta\sec^2\theta d\theta = \int 8u^2du = \frac{8u^3}{3} + C = \frac{8\sec^3\theta}{3} + C = \frac{\sqrt{t^2-4}^3}{3} + C$$

Note that this problem could also be solved, and arguably much more simply, by using a standard substitution, since $\sqrt{t^2-4}$, in addition to being a trigonometric form, is a composition of a square root with the expression x^2-4, and the derivative of x^2-4 appears in a modified form as a factor of the integrand, suggesting the substitution $u=x^2-4$, which gives $du=2xdx$, allowing us to rephrase the integral like so:

$$\int t\sqrt{t^2-4}dt = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2}+C = \frac{(t^2-4)^{3/2}}{3}+C$$

(b) **(10 points)** $\int x^3\sqrt{25-x^2}dx$.
The term $\sqrt{25-x^2}$ suggests a trigonometric substitution, and in particular a sine substitution. If we construct a right triangle with marked angle θ, hypotenuse length 5, opposite side length x, and adjacent side length $\sqrt{25-x^2}$, we find that $x=5\sin\theta$ and $\sqrt{25-x^2}=5\cos\theta$. Furthermore, for the substitution, we will need the differential $dt=5\cos\theta d\theta$. Then the above integral becomes

$$\int (5\sin\theta)^3(5\cos\theta)(5\cos\theta)d\theta = \int 5^5\sin^3\theta\cos^2\theta d\theta$$

Since the above integral has an odd number of multiplicative $sin\theta$ terms, judicious use of the identity $\sin^2\theta=(1-\cos^2\theta)$ will give an expression solely in terms of $\cos\theta$ save for a single multiplicative term $\sin\theta$:

$$\int 5^5\sin^3\theta\cos^2\theta d\theta = \int 5^5\sin\theta(1-\cos^2\theta)\cos^2\theta d\theta = \int 5^5(\cos^2\theta-\cos^4\theta)\sin\theta d\theta$$

This form is now ripe for the substitution $u=\cos\theta$, $du=-\sin\theta d\theta$, which will serve to simplify this integral:

$$\begin{aligned}\int 5^5(\cos^2\theta-\cos^4\theta)\sin\theta d\theta &= \int 5^5(u^2-u^4)(-du)\\ &= 5^5\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C\\ &= \frac{(5\cos\theta)^5}{5}-\frac{25(5\cos\theta)^3}{3}+C\\ &= \frac{\sqrt{25-x^2}^5}{5}-\frac{25\sqrt{25-x^2}^3}{3}+C\end{aligned}$$

6. **(20 points)** *Evaluate the following integrals:*

(a) **(10 points)** $\int \frac{t^2-4t-2}{t(t+1)(t-2)}dt$.
For this factorization into distinct linear terms, the appropriate decomposition is

$$\frac{t^2-4t-2}{t(t+1)(t-2)} = \frac{A}{t}+\frac{B}{t+1}+\frac{C}{t-2}$$

which, on multiplying by the common denominator, yields

$$\begin{aligned}t^2-4t-2 &= A(t+1)(t-2)+Bt(t-2)+Ct(t+1)\\ t^2-4t-2 &= A(t^2-t-2)+B(t^2-2t)+C(t^2+t)\\ t^2-4t-2 &= (A+B+C)t^2+(-A-2B+C)t-2A\end{aligned}$$

Comparing quadratic, linear, and constant terms on the left and right side of the above equation yields the system of equations

$$\begin{cases} A + B + C = 1 \\ -A - 2B + C = -4 \\ -2A = -2 \end{cases}$$

The last equation gives us $A = 1$ immediately; combined with the first two, we see that $B + C = 0$ and $-2B + C = -3$, which can be reduced to $B = 1$ and $C = -1$. Thus

$$\frac{t^2 - 4t - 2}{t(t+1)(t-2)} = \frac{1}{t} + \frac{1}{t+1} - \frac{1}{t-2}$$

so that

$$\int \frac{t^2 - 4t - 2}{t(t+1)(t-2)} dt = \int \frac{1}{t} + \frac{1}{t+1} - \frac{1}{t-2} dt = \ln|t| + \ln|t+1| - \ln|t-2| + C$$

(b) **(10 points)** $\int \frac{dx}{x^2+8x+25}$.
This is an irreducible quadratic; its denominator can thus be rephrased via completion of the square as a sum of two squars, which under appropriate division becomes an expression of the form u^2+1, which can be integrated using an implicit linear substitution:

$$\begin{aligned} \int \frac{dx}{x^2 + 8x + 25} &= \int \frac{dx}{x^2 + 8x + 16 + 9} \\ &= \int \frac{dx}{(x+4)^2 + 3^2} \\ &= \int \frac{\frac{1}{3^2} dx}{\left(\frac{x+4}{3}\right)^2 + 1^2} \\ &= \frac{1}{9} \int \frac{dx}{\left(\frac{x+4}{3}\right)^2 + 1^2} \\ &= \frac{1}{9} \cdot 3 \arctan \frac{x+4}{3} + C = \frac{1}{3} \arctan \frac{x+4}{3} + C \end{aligned}$$