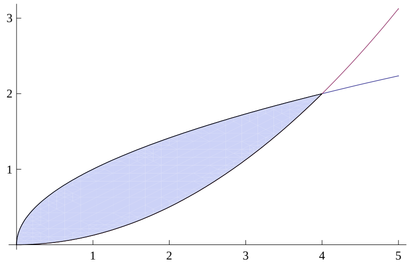
1. **(15 points)** *The region shown below is the area between the curves* $y = \sqrt{x}$ *and* $y = \frac{1}{8}x^2$. *Find the center of mass of this region.*



We must calculate the area, x-moment, and y-moment to find the center of mass.

The region is bounded on the left by $x = 0$ and on the right by $x = 4$. The upper curve is $y = \sqrt{x}$ and the lower curve is $y = \frac{1}{8}x^2$. Thus, the integral to calculate the area is

$$A = \int_0^4 \sqrt{x} - \frac{1}{8}x^2 dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{24}\Bigg]_0^4 = \left(\frac{8}{3/2} - \frac{64}{24}\right) - (0-0) = \frac{8}{3}$$

To calculate the x=moment, we multiply the integrand above by x to get

$$M_x = \int_0^4 x^{3/2} - \frac{1}{8}x^3 dx = \frac{x^{5/2}}{5/2} - \frac{x^4}{32}\Bigg]_0^4 = \left(\frac{32}{5/2} - \frac{256}{32}\right) - (0-0) = \frac{24}{5}$$

To calculate the y-moment, however, we need to use one half the difference of the squares of the upper and lower functions:

$$M_y = \int_0^4 \frac{1}{2}\left(\sqrt{x}\right)^2 - \frac{1}{2}\left(\frac{1}{8}x^2\right)^2 dx = \frac{1}{2}\int_0^4 x - \frac{x^4}{64} dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^5}{320}\right]_0^4 = \left(\frac{16}{4} - \frac{4^5}{5\cdot 64\cdot 2}\right) - (0-0) = \frac{12}{5}$$

Thus, the center of mass of the above region is $(\frac{M_x}{A}, \frac{M_y}{A})$

2. **(10 points)** *Consider the curve* $y = x^3 + 1$ *between the points* $(2, 9)$ *and* $(3, 28)$.

   (a) **(6 points)** *Construct, but do not evaluate, an integral representing the length of this curve.*

   The arclength expression $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ evaluates in this case to $\sqrt{1 + (3x^2)^2}$, so the arclength is

   $$\int_2^3 \sqrt{1 + 9x^4} dx$$

   (b) **(2 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the line* $y = 1$.

   Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1 + 9x^4}dx$, being spun around circles of radius $3x^2 - 1$, since the vertical distance between the line $y = 1$ and the point $(x, x^3 + 1)$ is simply $(x^3 + 1) - 1 = x^3$; thus the differential area traced out is $2\pi x^3\sqrt{1 + 9x^4}dx$, so the integral to compute the total surface area is

   $$\int_2^3 2\pi x^3 \sqrt{1 + 9x^4} dx$$

(c) **(2 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the y-axis.*

Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1+9x^4}dx$, being spun around circles of radius x, since the horizontal distance between the line $x = 0$ and the point $(x, x^3 + 1)$ is simply x; thus the differential area traced out is $2\pi x\sqrt{1+9x^4}dx$, so the integral to compute the total surface area is

$$\int_2^3 2\pi x\sqrt{1+9x^4}dx$$

3. **(15 points)** *Consider the function* $f(x) = \begin{cases} 0 \text{ for } x < 2 \\ \frac{64}{x^5} \text{ for } x \geq 2 \end{cases}$.

(a) **(6 points)** *Prove that $f(x)$ is a probability distribution function.*

A cursory inspection reveals that this function is non-negative throughout: 0 is non-negative everywhere, and $\frac{64}{x^5}$ is non-negative as long as $x > 0$. The critical property to demonstrate that this function is a probability distribution function is simply that $\int_{-\infty}^{\infty} f(x)dx = 1$. We can simplify this somewhat by ignoring the region on which $f(x)$ is zero, so that $\int_{-\infty}^{\infty} f(x) = \int_2^{\infty} f(x)dx$. We evaluate this as such:

$$\begin{aligned}\int_2^{\infty} f(x)dx &= \int_2^{\infty} \frac{64}{x^5}dx \\ &= \lim_{b\to\infty} \int_2^b \frac{64}{x^5}dx \\ &= \lim_{b\to\infty} \left.\frac{-16}{x^4}\right]_2^b \\ &= \lim_{b\to\infty} \frac{-16}{b^4} + \frac{16}{2^4} \\ &= 0 + 1 = 1\end{aligned}$$

(b) **(3 points)** *For a random variable X described by the above probability distribution function, find $P(X < 4)$.*

This probability will be simply $\int_2^4 f(x)dx$. It is techincally $\int_{-\infty}^4 f(x)dx$, but we can ignore the region over which the function is zero.

$$\int_2^4 f(x)dx = \int_2^4 \frac{64}{x^5}dx = \left.\frac{-16}{x^4}\right]_2^4 = \frac{-16}{4^4} + \frac{16}{2^2} = \frac{-1}{16} + 1 = \frac{15}{16}$$

(c) **(6 points)** *For a random variable X described by the above probability distribution function, find the average value of X.*

The expected value (or average value) of a probability distribution function $f(x)$ is $\int_{-\infty}^{\infty} xf(x)dx$. We may ignore locations where the integrand is zero, so this can be simplified to $\int_2^\infty xf(x)dx$:

$$\begin{aligned}\int_2^\infty xf(x)dx &= \int_2^\infty x\frac{64}{x^5}dx \\ &= \lim_{b\to\infty}\int_2^b \frac{64}{x^4}dx \\ &= \lim_{b\to\infty} \left.\frac{-64}{3x^3}\right]_2^b \\ &= \lim_{b\to\infty} \frac{-64}{3b^3} + \frac{64}{3\cdot 2^3} \\ &= 0 + \frac{64}{24} = \frac{8}{3}\end{aligned}$$

4. **(15 points)** *Perform the approximations shown below.*

   (a) **(5 points)** *Using Simpson's rule with $n = 4$, approximate $\int_0^8 \frac{1}{x^3+1}dx$. You need not arithmetically simplify your result.*

   We divide $[0, 8]$ into four intervals, each of length 2. Thus, our five sample points are $x = 0, 2, 4, 6, 8$, and the function evaluated at these points is $\frac{1}{0^3+1}$, $\frac{1}{2^3+1}$, $\frac{1}{4^3+1}$, $\frac{1}{6^3+1}$, and $\frac{1}{8^3+1}$. If we were to put these into Simpson's rule, we would get:

$$\int_0^8 \frac{1}{x^3+1}dx \approx \frac{2}{3}\left(\frac{1}{1} + \frac{4}{2^3+1} + \frac{2}{4^3+1} + \frac{4}{6^3+1} + \frac{1}{8^3+1}\right)$$

   This would, if calculated, be $\frac{21643864}{21707595} \approx 0.997$, which is not actually terribly close to the actual value of the integral value of approximately 1.201.

   (b) **(10 points)** *Using Euler's method with a step size of 2, approximate the value of y when $x = 16$ if $y = 5$ when $x = 10$ and $\frac{dy}{dx} = \frac{x}{y}$.*

   We have a differential equation whose slope (i.e. $\frac{dy}{dx}$) at each point is described by the function $m(x, y) = \frac{x}{y}$. We will be using Euler's method on this with $\Delta x = 2$ and initial point of $(x_0, y_0) = (10, 5)$. From this, we will calculate new positions x_1 and y_1.

$$x_1 = x_0 + \Delta x = 10 + 2 = 12$$
$$y_1 = y_0 + \Delta x m(x_0, y_0) = 5 + 2\cdot\frac{10}{5} = 9$$

   so the second point in our estimation of this curve is $(12, 9)$. We repeat Euler's method at this new point to find x_2 and y_2:

$$x_2 = x_1 + \Delta x = 12 + 2 = 14$$
$$y_2 = y_1 + \Delta x m(x_1, y_1) = 9 + 2\cdot\frac{12}{9} = \frac{35}{3}$$

   so the third point in our estimation of this curve is $(14, \frac{35}{3})$. We repeat Euler's method at this new point to find x_3 and y_3:

$$x_3 = x_2 + \Delta x = 14 + 2 = 16$$
$$y_3 = y_2 + \Delta x m(x_2, y_2) = \frac{35}{3} + 2\cdot\frac{14}{35/3}$$

   so when $x = 16$, we estimate that $y = \frac{35}{3} + \frac{12}{5}$, which need not be simplified.

5. **(15 points)** *Evaluate the following integrals, or if they cannot be evaluated, explain why not.*

(a) **(8 points)** $\int_{-4}^{3} \frac{1}{(x-2)^4} dx$.
This function has an infinite discontinuity at $x = 2$; the integral is thus improper and must be spliced around that point to actually be an evaluatable definite integral.

$$\begin{aligned}
\int_{-4}^{3} \frac{1}{(x-2)^4} dx &= \lim_{b\to 2^-} \int_{-4}^{b} \frac{1}{(x-2)^4} dx + \lim_{a\to 2^+} \int_{a}^{3} \frac{1}{(x-2)^4} dx \\
&= \lim_{b\to 2^-} \frac{1}{-3(x-2)^3}\Bigg]_{-4}^{b} + \lim_{a\to 2^+} \frac{1}{-3(x-2)^3}\Bigg]_{a}^{3} \\
&= \lim_{b\to 2^-} \frac{1}{-3(b-2)^3} - \frac{1}{-3(-6)^3} + \frac{1}{-3(a-2)^3} - \lim_{a\to 2^+} \frac{1}{-3(1)^3} \\
&= \frac{1}{-3(a-2)^3} - \frac{1}{-3(-6)^3} + \lim_{b\to 2^-} \frac{1}{-3(b-2)^3} - \lim_{a\to 2^+} \frac{1}{-3(1)^3}
\end{aligned}$$

Neither of the two limits above are evaluatable: as a and b approach 2, the denominators of the expressions shown approach zero, so the infinite discontinuity persists even after integration; thus this integral is divergent.

(b) **(7 points)** $\int_{-\infty}^{6} xe^{-x^2} dx$.
Using limits to rephrase this improper integral:

$$\begin{aligned}
\int_{-\infty}^{6} xe^{-x^2} dx &= \lim_{a\to -\infty} \int_{a}^{6} xe^{-x^2} dx \\
&= \lim_{a\to -\infty} \int_{x=a}^{x=6} -\frac{1}{2} e^u du \text{ for } u = -x^2; du = -2xdx \\
&= \lim_{a\to -\infty} -\frac{1}{2} e^u \Bigg]_{x=a}^{x=6} \\
&= \lim_{a\to -\infty} -\frac{1}{2} e^{-x^2} \Bigg]_{x=a}^{x=6} \\
&= \lim_{a\to -\infty} -\frac{1}{2} e^{-6^2} - \frac{1}{2} e^{-a^2} \\
&= \frac{e^{-6^2}}{2} - 0 = \frac{e^{-36}}{2}
\end{aligned}$$

6. **(20 points)** *Answer the following questions about the differential equation* $\frac{dy}{dx} = 3x^2\sqrt{1-y^2}$.

(a) **(5 points)** *Demonstrate without explicitly solving the differential equation that* $y = \sin(x^3)$ *is a solution.*
Using the chain rule, we know that $\frac{dy}{dx} = 3x^2 \cos(x^3)$. Evaluating the right side of the differential equation, we have

$$3x^2\sqrt{1-y^2} = 3x^2\sqrt{1-(\sin x^3)^2} = 3x^2\sqrt{1-\sin^2 x^3} = 3x^2\sqrt{\cos^2 x^3} = 3x^2 \cos x^3$$

Since the two sides of the differential equation are equal when $y = \sin(x^3)$ is substituted in, $y = \sin(x^3)$ is a solution of this differential equation.

(b) **(10 points)** *Find the general solution of the differential equation.*

Using separable methods, we rearrange this integral into

$$\begin{aligned}\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} &= 3x^2 \\ \int \frac{dy}{\sqrt{1-y^2}} &= \int 3x^2 dx \\ \arcsin y &= x^3 + C \\ y &= \sin(x^3 + C)\end{aligned}$$

Note that the specific solution given in part (a) is this general solution with $C = 0$.

(c) **(5 points)** *Using your general solution, find a solution to the differential equation subject to the initial condition that $y = 1$ when $x = 0$.*

Plugging $y = 1$ and $x = 0$ into the solution found above: $1 = \sin(0^3 + C)$, leading to $1 = \sin C$, so C could be (among other things) equal to $\frac{\pi}{2}$, leading to the specific solution $y = \sin(x^3 + \frac{\pi}{2})$

7. **(10 points)** *Find the general solution to the differential equation $\frac{dy}{dx} + 5y = 2x$.*

This is a linear differential equation, so we must determine an integrating factor given by the exponential function applied to the integral of the coefficient of y:

$$\rho = e^{\int 5dx} = e^{5x}$$

Multiplying the whole equation by this integrating factor, we see (or hope, at least) that the left side folds into a single derivative, which we can then integrate:

$$\begin{aligned}e^{5x}y' + 5e^{5x}y &= 2xe^{5x} \\ \frac{d}{dx}\left(e^{5x}y\right) &= 2xe^{5x} \\ \int \frac{d}{dx}\left(e^{5x}y\right)dx &= \int 2xe^{5x}dx \\ e^{5x}y &= 2\int xe^{5x}dx\end{aligned}$$

The interal on the right side of this equation is calculated using integration by parts with $u = x$, $dv = e^{5x}dx$, so $du = dx$, and $v = \frac{1}{5}e^{5x}$:

$$\begin{aligned}e^{5x}y &= 2\int xe^{5x}dx \\ &= 2\left(x\cdot\frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}dx\right) \\ &= 2\left(\frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}\right) + C \\ \therefore y &= \frac{2}{5}x - \frac{1}{25} + \frac{C}{e^{5x}}\end{aligned}$$