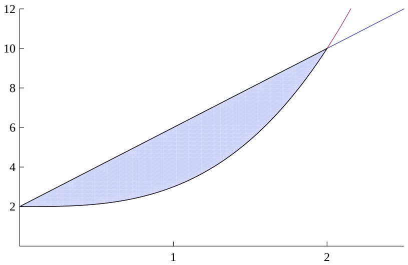
1. **(8 points)** *Find the center of mass of the region bounded by the curves $y = x^3 + 2$ and $y = 4x + 2$. Expressions need not be arithmetically reduced.*



To find the center of mass, we need first find the area A, x-moment M_x, and y-moment M_y:

$$A = \int_0^2 (4x+2) - (x^3+2)dx = \int_0^2 4x - x^3 = 2x^2 - \frac{x^4}{4}\Big]_0^2 = (8-4) - (0-0) = 4$$

$$M_x = \int_0^2 x[(4x+2) - (x^3+2)]dx = \int_0^2 4x^2 - x^4 = \frac{4x^3}{3} - \frac{x^5}{5}\Big]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$$

$$\begin{aligned} M_y &= \int_0^2 \frac{1}{2}[(4x+2)^2 - (x^3+2)^2]dx = \int_0^2 -\frac{1}{2}x^6 - 2x^3 + 8x^2 + 8x dx \\ &= -\frac{x^7}{14} - \frac{x^4}{2} + \frac{8x^3}{3} + 4x^2\Big]_0^2 = -\frac{64}{7} - 8 + \frac{64}{3} + 16 = \frac{424}{21} \end{aligned}$$

So the center of mass is $\left(\frac{M_x}{A}, \frac{M_y}{A}\right) = \left(\frac{16}{15}, \frac{108}{21}\right)$.

2. **(8 points)** *Let* $f(x) = \begin{cases} 0 \text{ for } x < 1 \\ \dfrac{2}{x^3} \text{ for } x \geq 1 \end{cases}$

   (a) **(4 points)** *Verify that $f(x)$ is a probability distribution function.*

   A cursory inspection reveals that this function is non-negative throughout: 0 is non-negative everywhere, and $\frac{2}{x^3}$ is non-negative as long as $x > 0$. The critical property to demonstrate that this function is a probability distribution function is simply that $\int_{-\infty}^{\infty} f(x)dx = 1$. We can simplify this somewhat by ignoring the region on which $f(x)$ is zero, so that $\int_{-\infty}^{\infty} f(x) = \int_1^{\infty} f(x)dx$. We evaluate this as such:

$$\begin{aligned}\int_1^\infty f(x)dx &= \int_1^\infty \frac{2}{x^3}dx \\ &= \lim_{b\to\infty}\int_1^b \frac{2}{x^3}dx \\ &= \lim_{b\to\infty}\int_1^b \frac{2}{x^3}dx \\ &= \lim_{b\to\infty} \left.\frac{-1}{x^2}\right]_1^b dx \\ &= \lim_{b\to\infty} \frac{-1}{b^2} + \frac{1}{1^2}dx \\ &= 1\end{aligned}$$

(b) **(4 points)** *For a random variable X described by the above probability distribution function, find $P(X \geq 10)$.*

We proceed as above but calculating $P(X \geq 10)$, which is $\int_{10}^\infty f(x)$:

$$\begin{aligned}\int_{10}^\infty f(x)dx &= \int_{10}^\infty \frac{2}{x^3}dx \\ &= \lim_{b\to\infty}\int_{10}^b \frac{2}{x^3}dx \\ &= \lim_{b\to\infty} \frac{-1}{b^2} + \frac{1}{10^2}dx \\ &= \frac{1}{100}\end{aligned}$$

Thus, this random variable will only have a value greater than 10 in one out of every hundred tests.

3. **(8 points)** *Answer the following questions about the differential equation $\frac{dy}{dx} = y^2$.*

(a) **(4 points)** Verify that $y = \frac{1}{5-x}$ is a solution to this differential equation.

We evaluate each side, substituting $\frac{1}{5-x}$ in for y:

$$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{5-x} = \frac{d}{dx}(5-x)^{-1} = (-1)\left[-(5-x)^{-2}\right] = (5-x)^{-2}$$

$$y^2 = \left(\frac{1}{5-x}\right)^2 = \frac{1}{(5-x)^2} = (5-x)^{-2}$$

Since these evaluations are identical, the assignment $y = \frac{1}{5-x}$ indeed satisfies the differential equation.

(b) **(4 points)** Using Euler's method, if $y = 2$ when $x = 1$, estimate the value of y when $x = 1.2$, using a step size of 0.1.

We have a differential equation whose slope (i.e. $\frac{dy}{dx}$) at each point is described by the function $m(x, y) = y^2$. We will be using Euler's method on this with $\Delta x = 0.1$ and initial point of $(x_0, y_0) = (1, 2)$. From this, we will calculate new positions x_1 and y_1.

$$x_1 = x_0 + \Delta x = 1 + 0.1 = 1.1$$

$$y_1 = y_0 + \Delta x m(x_0, y_0) = 2 + 0.1(2^2) = 2.4$$

so the second point in our estimation of this curve is $(1.1, 2.4)$. We repeat Euler's method at this new point to find x_2 and y_2:

$$x_2 = x_1 + \Delta x = 1.1 + 0.2 = 1.2$$

$$y_2 = y_1 + \Delta x m(x_1, y_1) = 2.4 + 0.1(2.4^2) = 2.976$$

so when $x = 1.2$, we estimate that $y = 2.976$.

Note: the actual solution to this differential equation with initial condition, as discussed in class, is $y = \frac{1}{\frac{3}{2} - x}$; when $x = 1.2$ the actual value of y is thus actually $\frac{10}{3}$, which is not actually all that close to 2.976.