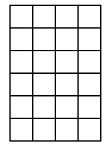
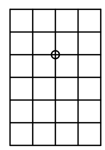
1. **(12 points)**

   (a) **(4 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following two-dimensional grid?*



A path through this grid consists of 6 upwards steps and 4 steps to the right. Thus, we may associate any given gridwalk with a series of 10 instuctions, 4 of which are right-steps and the rest of which are up-steps. The number of distinct walks is thus the number of ways to choose 4 positions from 10, or $\binom{10}{4} = 210$.

   (b) **(8 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following two-dimensional grid which pass through the marked point?*



Such a path is a concatenation of two sub-paths: one from the lower left corner to the marked point, and one from the marked point to the upper right corner. The former is a walk consisting of 4 up-steps and 2 right-steps, and using the argument above, this can be achieved in $\binom{6}{2}$ ways; the latter path consists of 2 up-steps and 2 right-steps, so there are $\binom{4}{2}$ ways to do this. Thus, since a full path is a composition of a choice of each of these paths, there are $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$ possible walks total.

2. **(12 points)** *You find that you need to buy 22 hats. The hat shop has 4 different varieties of hat: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.*

   (a) **(4 points)** *One example of a hat purchase would be: 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes. How many different possible ways are there for you to purchase 22 hats?*

If we define the variables $x_1, \ldots, x_4$ to be respectively the number of stetsons, berets, stovepipes, and pillboxes, then a hat purchase is equivalent to a selection of non-negatve values for the variables such that $x_1 + x_2 + x_3 + x_4 = 22$. We know that this is $\binom{22+4-1}{4-1} = \binom{25}{3} = 2300$; alternatively, we could consider the number of ways to place 3 dividers among 22 hats, so that the hats are partitioned into 4 (possibly empty) groups, which will be declared to represent different styles of hat. This would be enumerated with $\binom{22+3}{3}$, as above.

   (b) **(2 points)** *Your friend wants several identical hats and is not sure which ones you bought. What is the largest number of identical hats he can be certain to be able to borrow from you?*

Consider types of hats as pigeonholes, and individual hats as pigeons. Your friend does not know what you are buying; i.e. how the pigeons are distributed among the holes, but since there are 4 pigeonholes and more than $4 \cdot 5$ pigeons, the pigeonhole principle allows him to be certain that you have at least $5 + 1 = 6$ of one type.

(c) **(6 points)** *Suppose you want to select your lot of 22 hats so that there are at least 3 hats of each type. How many ways are there to fulfill these instructions?*

This situation is as in the first part of this problem, except instead of constraining each x_i to be non-negative, we constrain each to be at least 3; we can transform this lower bound into zero by letting each $y_i = x_i - 3$, inducing a natural bijection between integer solution sets to

$$x_1 + x_2 + x_3 + x_4 = 22 \quad x_i \geq 3$$

and

$$(y_1 + 3) + (y_2 + 3) + (y_3 + 3) + (y_4 + 3) = 22 \quad y_i \geq 0$$

The latter simplifies to determining nonnegative solutions to $y_1+y_2+y_3+y_4 = 10$, which can be done in any of $\binom{10+3}{3} = 286$ ways.

Alternatively, the above solution can be justified by pre-emptively assigning 3 hats of each type, leaving 10 hats left to be assigned.

3. **(12 points)** *A game is played with a fifty-card deck consisting of cards in the 5 suits of acorns, hearts, leaves, bells, and trumps, numbered 1 to 10.*

(a) **(4 points)** *How many 5-card hands consist of one card in each suit, with no restrictions on numbers?*

Our hand has no order, but individual cards may be distinguished by virtue of being different suits. Consider the card in the suit of acorns: it has 10 possible numbers. Likewise, the card in the suit of hearts has 10 possible numbers, as do the cards in leaves, bells, and trumps. A hand is thus uniquely determined by a process of 5 decisions, each of which can be resolved in 10 different ways, so there are $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000$ possible hands.

(b) **(8 points)** *How many 5-card hands have two pairs (two cards in each of two different numbers, and a fifth card in a different number than either)?*

Let us consider the features of this hand: there are two pairs, which have no intrinsic order, and one leftover card which is distinguished from these two. We must choose two distinct numbers to appear in the pairs, and since there is no intrinsic order, there are $\binom{10}{2}$ ways to do so. The number for the leftover card can be any nubmer which does not appear in the pairs – we will thus have 8 choices for this card. We may thus determine the numbers appearing in our hand any of $\binom{10}{2} \cdot 8$ ways (we could also express this as $\binom{10}{2,1,7}$ to represent choosing from a list of 10 numbers two numbers to designate as pairs, one to designate as a single, and seven to designate as not appearing in this hand).

Now we must choose suits: Here we can distinguish between the pairs, since the pairs are distinguished by virtue of having distinct numbers chosen (e.g. we could assign suits to the lower-numbered pair, and then the higher-numbered pair). For each pair there are $\binom{4}{2}$ ways to choose suits, and there are $\binom{4}{1}$ ways to choose suits for the singleton. Thus, multiplying the nubmer of ways to choose numbers by the number of ways to choose suits, we get $\binom{10}{2}\binom{8}{1}\binom{4}{2}^2\binom{4}{1} = 56160$.

4. **(12 points)**

(a) **(3 points)** *Evaluate* $1 + 6 + 11 + 16 + \cdots + 191 + 196$.

Let $S = 1+6+11+16+\cdots+191+196$. Then, let us reverse the sum and add together the equations below to observe:

$$\begin{aligned} S &= \quad 1 + \quad 6 + \ 11 + \ 16 + \cdots + 191 + 196 \\ S &= 196 + 191 + 186 + 181 + \cdots + \quad 6 + \quad 1 \\ 2S &= 197 + 197 + 197 + 197 + \cdots + 197 + 197 \end{aligned}$$

so $2S = 40 \cdot 197$, and thus $S = \frac{40 \cdot 197}{2} = 3940$.

(b) **(9 points)** *Prove by induction that* $1+2+4+8+16+\cdots+2^n = 2^{n+1}-1$ *for every integer* $n \geq 1$.

For the base step, let us note that $1+2 = 3 = 2^2-1$, so the above statement is clearly true for $n = 1$ (in fact, it is also true for $n = 0$, and that could be used as the base step with equal validity). Now, given the inductive hypothesis $1+2+4+8+16+\cdots+2^n = 2^{n+1}-1$, we wish to show that $1+2+4+8+16+\cdots+2^{n+1} = 2^{n+2}-1$. We start with the inductive hypothesis, and perform algebra until we demonstrate our desired consequence:

$$\begin{aligned} 1+2+4+8+16+\cdots+2^n &= 2^{n+1}-1 \\ 1+2+4+8+16+\cdots+2^n+2^{n+1} &= 2^{n+1}-1+2^{n+1} \\ 1+2+4+8+16+\cdots+2^n+2^{n+1} &= 2\cdot 2^{n+1}-1 \\ 1+2+4+8+16+\cdots+2^n+2^{n+1} &= 2^{n+2}-1 \end{aligned}$$

5. **(12 points)**

(a) **(4 points)** *Determine the coefficient of* x^2 *in* $(2x+1)(x+4)^5$.

There are two possible combinations yielding an x^2 term: either $(x+4)^5$ and $(2x+1)$ both contribute a factor of x, or $(x+4)^5$ contributes a factor of x^2 and $(2x+1)$ contributes no x-terms. The coefficient of x in the expansion of $(x+4)^5$ is $4^4 \cdot \binom{5}{1}$, so the first combination contributes the x^2 term $(2x)\left(4^4\binom{5}{1}x\right) = 2\cdot 4^4\binom{5}{1}x^2$. The coefficient of x^2 in the expansion of $(x+4)^5$ is $4^3 \cdot \binom{5}{2}$, so the second combinaiton contributes the x^2 term $(1)\left(4^3\binom{5}{3}x^2\right)$. Thus, our total coefficient of x^2 is $2\cdot 4^4\binom{5}{1} + 4^3\binom{5}{2} = 3200$.

(b) **(4 points)** *Determine the coefficient of* xyz^3 *in* $(6x-3y+2z)^5$.

We know that the multinomial expansion yields the term $\binom{5}{3,1,1}(6x)(-3y)(2z)^3$, so segregating out the constants yields $6(-3)(2^3)\binom{5}{3,1,1}xyz^3 = -2880xyz^3$ so the coefficient is -2880.

(c) **(4 points)** *Simplify the expression*

$$\binom{n}{0} + 3\binom{n}{1} + 9\binom{n}{2} + 27\binom{n}{3} + \cdots + 3^n\binom{n}{n}$$

We know that

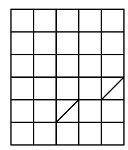
$$(1+x)^n = \binom{n}{0} + x\binom{n}{1} + x^2\binom{n}{2} + x^3\binom{n}{3} + \cdots + x^n\binom{n}{n}.$$

Substituting in $x = 3$ yields the expression asked about on the right side of the equation:

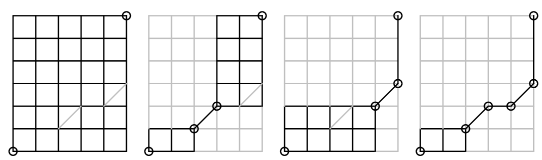
$$(1+3)^n = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + \cdots + 3^n\binom{n}{n}.$$

So this sum is 4^n.

6. **(6 point bonus)** *How many paths consisting of steps to the north, east, and northeast are possible from the southwest to northeast corner of the following grid?*



We can solve this problem by counting disjoint groups of paths classified by their diagonal-edge utilization. Below are four subgrids of the given grid, with indicated waypoints necessary to visit:



Let us start by considering paths not using any diagonals, as in the first image shown. This is clearly possible in $\binom{11}{5}$ ways. The second possibility, exploiting the lower left diagonal, must pass through the waypoints shown on the the second image. Frim the first to the second marked point, there are $\binom{3}{1}$ paths; only one path along the diagonal, and $\binom{6}{2}$ paths on the other side of the diagonal. Thus this subgrid can be walked in $\binom{3}{1} \cdot 1 \cdot \binom{6}{2}$ ways. Likewise, the third and fourth diagonal-utilizing grids contribute $\binom{6}{2} \cdot 1 \cdot \binom{3}{0}$ and $\binom{3}{1} \cdot 1 \cdot \binom{1}{0} \cdot 1 \cdot \binom{3}{0}$ walks respectively. Adding these up, we get a total of

$$\binom{11}{5} + \binom{3}{1}\binom{6}{2} + \binom{6}{2}\binom{3}{0} + \binom{3}{1}\binom{1}{0}\binom{3}{0} = 525$$