1. **(12 points)** *Answer the following questions.*

   (a) **(6 points)** *How many arrangements of the word "MURMUR" are there in which no two consecutive letters are the same?*

   Let X consist of all permutations of the letters in MURMUR; let A_M, A_U, and A_R consist of those permutations in which the Ms, Us, and Rs respectively are adjacent. We want to determine how many rearrangements are in none of these sets, so we seek $|\overline{A_M \cup A_U \cup A_R}|$. $|X|$ enumerates all arrangements of 2 items from three different classes, so, using the multinomial coefficient, $|X| = \binom{6}{2,2,2}$. Using the trick of considering MM as a monolithic element, we can see that A_M contains arrangements of two Us, two Rs, and one MM, so $|A_M| = \binom{5}{1,2,2}$; $|A_U|$ and $|A_R|$ are equivalent. We can similarly construct each $|A_i \cap A_j|$ by considering two monolithic pairs and two identical letters of the third type (e.g. MM, UU, and two Rs in the case of $|A_M \cap A_U|$), so $|A_i \cap A_j| = \binom{4}{1,1,2}$, and lastly, $|A_M \cap A_U \cap A_R| = \binom{3}{1,1,1}$, since we have three distinct monolithic pairs. Thus:

   $$|\overline{A_M \cup A_U \cup A_R}| = \binom{6}{2,2,2} - 3\binom{5}{1,2,2} + 3\binom{4}{1,1,2} - \binom{3}{1,1,1}$$

   which evaluates to 30 but need not be computed.

   (b) **(6 points)** *We have 15 distinct books which we would like to divide among 3 indistinguishable boxes. How many ways are there to do so if we require that each box be nonempty?*

   This is the enumeration statistic for distinct balls in nondistinct greedy boxes; if you remember the formula for Stirling numbers, you know that this is

   $$S(15,3) = \frac{3^{15} - \binom{3}{1}2^{15} + \binom{3}{2}1^{15} - \binom{3}{3}0^{15}}{3!}$$

   However, to derive this from first principles, we start by determining the number of divisions of books among distinguishable boxes: this is essentially the number of surjections from a 15-element set to a 3-element set. Let X be the set of all placement of labeled books in labeled boxes; let A_i be the set of placements leaving the ith box empty. The number of free placements of 15 objects into 3 categories gives $|X| = 3^{15}$; free placements into only 2 categories gives that each $|A_i| = 2^{15}$, and likewise $|A_i \cap A_j| = 1^{15}$. $A_1 \cap A_2 \cap A_3$ is empty, since we cannot leave all three boxes empty and still place the fifteen books somewhere.

   Then, to find the number of placements leaving no boxes empty, we calculate

   $$|\overline{A_1 \cup A_2 \cup A_3}| = |X| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| = 3^{15} - \binom{3}{1}2^{15} + \binom{3}{2}1^{15} - 0$$

   This, however, is the number of distributions to *labeled* boxes. To determine the number of distributions to unlabeled boxes, we must consider the 3! possible permutations of labels, and divide by 3! to get $\frac{3^{15} - \binom{3}{1}2^{15} + \binom{3}{2}1^{15} - 0}{6}$ possible distributions (which evaluates to 2375101, but doing so would require calculations beyond the scope of this exam).

2. **(12 points)** *Suppose we are partitioning a number into no more than* 2 *parts each of size* 2*,* 3*, or* 4*. For instance,* 8 *has* 3 *such partitions:* $4 + 4$, $4 + 2 + 2$, $3 + 3 + 2$.

   (a) **(6 points)** *Determine a generating function for the number of unordered partitions of* n *into parts of size* 2*,* 3*, or* 4*, with no more than* 2 *parts of each size. You need not expand the generating function algebraically.*

   We may have a total of 0, 2, or 4 among our parts of size 2 (by having zero, one, or two of them). Thus, a generating function to record the total contributed by parts of size 2 is $1 + z^2 + z^4$; likewise, since parts of size 3 can contribute 0, 3, or 6, they have generating function $1 + z^3 + z^6$, and similarly parts of size 4 have contributions recorded by the generating function $1 + z^4 + z^8$. The contributions by all three of these aspects of a partition can be determined by multiplying to get the overall generating function

   $$(1 + z^2 + z^4)(1 + z^3 + z^6)(1 + z^4 + z^8)$$

   (b) **(2 points)** *What will the highest exponent appearing in the expansion of the above generating function be? What does it signify?*

   The above can be seen to have highest-degree term $z^4 \cdot z^6 \cdot z^8 = z^{18}$. This means that the coefficient of z^n foe $n > 18$ is zero; that is to say, we cannot partition any number greater than 18 in this manner (which is somewhat obvious, since using each part as much as possible yields $2 + 2 + 3 + 3 + 4 + 4 = 18$, and we cannot partition a number higher than that).

   (c) **(4 points)** *Demonstrate that the transpose of a partition meeting the conditions described above will have fewer than* 5 *parts and no part of size larger than* 6*.*

   A partition according to the scheme given in this problem can have no more than 6 parts (since we can have at most 2 each from 3 different sizes) and no part of size greater than 4. The transposition operation swaps the largest number appearing in a partition and the number of parts appearing in the partition. Thus, since a partition meeting the above conditions has no part of size greater than 4, its transpose will have no more than 4 parts; since a partition meeting the above conditions has no more than 6 parts, its transpose will have no part of size greater than 6.

3. **(12 points)** *We have a row of several demarked small parking spaces on the side of the street into which we place vehicles of various types. A compact, a sedan, or a smartcar fits into a single space. An SUV, a van, a station wagon, or a pickup truck requires two spaces.*

   (a) **(6 points)** *Find a recurrence relation, with initial conditions, describing the number of configurations of various types of cars filling* n *parking spaces.*

   Let a_n represent the number of configurations of cars in n consecutive spaces. Then, for $n \geq 2$, consider the very first car in line: it could be a compact, sedan, or smartcar, leaving $n - 1$ empty spaces after to be filled in any of a_{n-1} ways. There are thus $3a_{n-1}$ such configurations (3 choices of first car, a_{n-1} ways to fill the remaining spaces). On the other hand, the first car may be an SUV, van, station wagon, or pickup truck, in which case there are only $n-2$ remaining empty spaces, which can be filled in a_{n-2} ways. This case thus accounts for $4a_{n-2}$ configurations, since there are 4 choices of first car. Putting these two cases together, we see that there are $3a_{n-1}+4a_{n-2}$ configurations to fill n parking spaces, so $a_n = 3a_{n-1} + 4a_{n-2}$. The initial conditions can be shown to be $a_0 = 1$ (since zero spaces can only be filled one way, with no cars), and $a_1 = 3$ (since one space may contain any of the three small cars).

(b) **(6 points)** *Solve the above-determined recurrence relation to determine a closed form for the number of configurations.*

The characteristic equation of $a_n = 3a_{n-1} + 4a_{n-2}$ is $x^2 - 3x - 4 = 0$, which has roots -1 and 4, so the general solution is $a_n = k_1(-1)^n + k - 2(4^n)$. Plugging in the known values $a_0 = 1$ and $a_1 = 3$, we get:

$$\begin{cases} 1 = k_1 + k_2 \\ 3 = -k_1 + 4k_2 \end{cases}$$

which has solutions $k_2 = \frac{4}{5}$ and $k_1 = \frac{1}{5}$, so

$$a_n = \frac{(-1)^n + 4(4^n)}{5}$$

4. **(12 points)**

(a) **(4 points)** *Find (but do not expand) an exponential generating function for the number of ternary sequences containing an even number of* 0*s, fewer than four* 1*s, and at least two* 2*s.*

Here order matters, so we need an exponential generating function, formed by multiplying the exponential generating functions for appearances of each digit. The exponential generating function representing appearances of zeroes is thus $1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ (which may, but need not, be represented as $\frac{e^z+e^{-z}}{2}$ or $\cosh z$). Appearance of ones is represented by $1 + z + \frac{z^2}{2} + \frac{z^3}{6}$, and appearance of twos is given by $\frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$, which can, but need not, be represented by $e^z - 1 - z$. Our overall generating function is thus

$$\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)\left(\frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right)$$

or

$$\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)(e^z - 1 - z)\cosh z$$

(b) **(4 points)** *Find a generating function describing the number of ways to put n indistinguishable items into 5 distinct boxes, with at least one and no more than five items per box.*

Since one can place between one and five indistinguishable items into the first box, the generating function for item placement in the first box is $z + z^2 + z^3 + z^4 + z^5$.

Since the same function suffices for the other four boxes, the product of all five generating functions is

$$(z + z^2 + z^3 + z^4 + z^5)^5$$

(c) **(6 points)** *Using your generating function, find the number of ways to distribute thirteen items in the manner described in part (b).*

We want the coefficient of z^{13} in the above, and we get it through a series of algebraic manipulations:

$$\begin{aligned}
(z + z^2 + z^3 + z^4 + z^5)^5 &= z^5(1 + z + z^2 + z^3 + z^4)^5 \\
&= z^5 \left(\frac{1 - z^5}{1 - z}\right)^5 \\
&= z^5(1 - z^5)^5 \frac{1}{(1 - z)^5} \\
&= z^5(1 - z^5)^5 \sum_{n=0}^{\infty} \binom{n + 5 - 1}{5 - 1} z^n \\
&= (z^5 - 5z^{10} + 10z^{15} - 10z^{20} + 5z^{25} - z^{30}) \sum_{n=0}^{\infty} \binom{n + 4}{4} z^n
\end{aligned}$$

We wish to determine all the z^{13} terms in the above product. There are two terms of the product which will contribute: $z^5 \cdot \binom{8+4}{4} z^8$, and $-5z^{10} \cdot \binom{3+4}{4} z^3$. These terms will form the single term $\left[\binom{12}{4} - 5\binom{7}{4}\right] z^{13}$ in the full expansion, so the number of distributions is the coefficient $\binom{12}{4} - 5\binom{7}{4}$.

5. **(6 point bonus)** *Find a generating function for the number of partitions of n such that if the partition contains a part of size k, it also contains an part of every size less than k (e.g., 6 can be so divided 4 ways, into only 1s: $1 + 1 + 1 + 1 + 1 + 1$, 1s and 2s: $1 + 1 + 1 + 1 + 2$ or $1 + 1 + 2 + 2$, or 1s, 2s, and 3s: $1 + 2 + 3$.)*

In the Ferrer's diagram for such a partition, we would have rows such that subsequent rows differ in length by 1 or 0. If we perform the transposition bijection on this, we would get a diagram in which subsequent columns differ in length by 1 or 0. The length of the ith column is the number of parts of size i or greater; thus the difference in length between the $i-1$th and ith column would be the number of parts of size exactly i; thus, the transpose of a partition satisfying the condition above is a partition in which each number appears either zero or one times; that is, the number of partitions of n into distinct parts (this can be illustrated by transposing the partitions of 6 given above). The generating function for this sequence is:

$$(1 + z)(1 + z^2)(1 + z^3)(1 + z^4) \cdots$$