1. **(12 points)** *Computationally, a vector is simply a list of numbers. We may represent an n-dimensional vector $\vec{a}$ as a list of n coordinates $(a_1, a_2, a_3, \ldots, a_n)$.*

   (a) **(9 points)** *Write an algorithm to compute the dot product of the vectors $\vec{a}$ and $\vec{b}$. Recall that a dot product of two vectors is the sum of the products of each coordinate.*

   Input: sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$
   Output: number c
   set c **to** 0;
   set i **to** 0;
   while $i \leq n$ **do**
   | **set** c **to** $c + a_i b_i$;
   | **set** i **to** $i + 1$;
   return determined value c;

   (b) **(3 points)** *Justify and state a good asymptotic bound in big-O notation on the number of steps taken by your algorithm.*

   The innermost set of instructions (adding a product to c) takes constant time, or $O(1)$ time. However, this instruction is repeated when i is 1, when i is 2, when i is 3, and so forth up to when i is n, so the time taken by performing this set of instructions n times is $n \cdot O(1)$, or the linear time $O(n)$.

2. **(12 points)** *Tyrell Corporation is commissioning the construction of a tower built from blocks of various materials. A section in brick or sandstone each cost one million woolongs per section; granite, slate, and marble cost two million per section.*

   (a) **(5 points)** *Find a recurrence relation and initial conditions for the number a_n of different towers that could be built with a budget of n million woolongs.*

   Let us consider how a tower costing n million woolongs could be built from smaller towers. We could add an inexpensive block to a tower costing $n - 1$ million, or we could add one of our premium blocks to a tower costing $n - 2$ million. There are a_{n-1} towers costing $n - 1$ million, and 2 types of inexpensive blocks we could add on top, so we can construct $2a_{n-1}$ towers in such a manner, while there are a_{n-2} towers costing $n - 2$ million, and 3 types of premium blocks we could add on top, so there are $3a_{n-2}$ towers we could construct according to the second description above. Thus, we could construct a tower costing n million in any of $2a_{n-1} + 3a_{n-2}$ ways. Thus, $a_n = 2a_{n-1} + 3a_{n-2}$. The initial conditions are $a_0 = 1$, since the only tower that can be build with no money is an empty tower, and $a_1 = 2$, since with one million woolongs one could construct a single block in either of the inexpensive materials.

   (b) **(5 points)** *Solve the recurrence relation determined above.*

   The characteristic equation of the recurrence relation above is $x^2 = 2x + 3$, which has roots $x = -1$ and $x = 3$, as can be determined by solving the quadratic. Thus, the general solution to the recurrence $a_n = 2a_{n-1} + 3a_{n-2}$ is $a_n = k_1 3^n + k_2(-1)^n$. The initial conditions are necessary to determine k_1 and k_2. Looking at the particular cases of $n = 0$ and $n = 1$, the above equation becomes:

$$\begin{cases} 1 = k_1 + k_2 \\ 2 = 3k_1 - k_2 \end{cases}$$

Solving this system of equations gives $k_1 = \frac{3}{4}$ and $k_2 = \frac{1}{4}$, so $a_n = \frac{3(3^n)+(-1)^n}{4}$.

(c) **(2 points)** *Suppose that we have purchased four blocks each of brick and sandstone, and two blocks each of granite, slate, and marble. Write (but do not simplify) an exponential generating function for the number b_n of different towers of height n that could be built with these materials.*

Let us write an exponential generating function for utilization of each material: by the multiplication principle for exponential generating functions, multiplying these will count all arrangements of different sections in order. For instance, there are five ways to use brick alone: we can build a tower of height zero, one, two, three, or four, so classifying these towers by height, this corresponds to the generating function $(1)\frac{z^0}{0!} + (1)\frac{z^1}{1!} + (1)\frac{z^2}{2!} + (1)\frac{z^3}{3!} + (1)\frac{z^4}{4!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$.The selection function for towers made of sandstone alone will be identical; since we only have two blocks of each of the premium materials, their generating functions will be $1 + z + \frac{z^2}{2}$, so multiplying together all these generating functions for constituent parts of the tower, we get

$$\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right)^2 \left(1 + z + \frac{z^2}{2}\right)^3$$

3. **(12 points)** *Let T be a tree with n vertices and maximum degree Δ.*

(a) **(6 points)** *Prove that the number of leaves (vertices of degree 1) in T is at least Δ.*

This can be comparatively easily proven by induction on the number of vertices n in the graph (and in other ways as well). The base case $n = 1$ is trivial, since $\Delta = 0$ on a one-vertex graph. Now, let us assume that the above statement is true on every n-vertex tree, and try to prove it on a tree of $n+1$ vertices. Consider T with $n+1$ vertices. Since T is a nontrivial tree, it has a leaf u with a single incident edge e and adjacent vertex v. Now consider T', the subtree consisting of all vertices and edges of T except u and e. Since the only removed edge is e, the degree of every vertex is the same in T and T' except for v, whose degree may be reduced by one. Thus, the maximum degree in T' is either Δ (if it occurred at some vertex other than v) or $\Delta - 1$ (if v happened to be the vertex of highest degree). By our inductive hypothesis, T' has $\Delta - 1$ leaves if v is a vertex of high degree, and Δ leaves otherwise. T has all the same leaves as T' except for the addition of u and possible removal of v, so T will have Δ leaves.

(b) **(6 points)** *Prove that there is a set of at least $\frac{n}{2}$ vertices in T such that no two of them are adjacent.*

The vertices of T can be two-colored by the following algorithm: color a vertex red; color its neighbors green; color the neighbors of any green vertex red and vice versa; repeat until all vertices are colored. In this manner we can color all the vertices red or green so that no two of the same color are adjacent. By the pigeonhole, at least half of the vertices must be one or the other color, so we may select whichever color class is larger to satisfy the conditions of this problem.

4. **(12 points)** *Jill is playing a game on a board divided into the five continents of America, Europe, Asia, Australia, and Africa. She has twenty-five identical tokens to play on the various regions and must play at least one token on each region. In addition, she may not play more than 5 tokens on any of Australia, Europe, or Africa.*

(a) **(4 points)** *Using traditional combinatorial methods, determine how many ways there are for her to place all her pieces.*

Let us pre-emptively place the requisite token in each territory, depleting our stock by 5; thus, this problem is identical to the question of how to place twenty tokens in the territory so that Australia, Europe, and Africa do not have more than 4 each.

Let X consist of all possible distributions of the 20 remaining tokens without restriction; Let A_1, A_2, and A_3 represent those distributions which give Australia, Europe, or Africa respectively more than 4 tokens.

Using the distribution statistic for undistinguished objects to distinct recipients (or the "balls and walls" argument used to justify that statistic) we may see that $|X| = \binom{24}{4}$.

Now, A_1 (and by analogy the other A_is) consists of those arrangements in which Australia (or some other territory, in the case of the other A_i) has at least 5 tokens. If we assign Australia an allotment of 5 tokens, then there are 15 left to be distributed; this can be done in $\binom{19}{4}$ ways, so $|A_i| = \binom{19}{4}$.

If we consider the intersection of some A_i and A_j, this would consist of those arrangements in which two territories have at least 5 tokens each; pre-emptively alloting these, we see that there are 10 tokens left to be placed, which can be done in $\binom{14}{4}$ ways, so $|A_i \cap A_j| = \binom{14}{4}$, and likewise $A_1 \cap A_2 \cap A_3$ consists of those arrangements produced by pre-emptively placing 15, then placing the remaining 5 freely, so $|A_1 \cap A_2 \cap A_3| = \binom{9}{4}$.

Thus, by inclusion-exclusion,

$$|\overline{A_1 \cup A_2 \cup A_3}| = |X| - 3|A_i| + 3|A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| = \binom{24}{4} - 3\binom{19}{4} + 3\binom{14}{4} - \binom{9}{4} = 1875$$

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(b) **(4 points)** *Construct a generating function for the number of ways to place n tokens on the board following the above rules. Your answer need not be algebraically simplified.*

America and Asia can get any number of tokens except zero, and each number of tokens can only be achieved in one way, so their generating functions are $(z + z^2 + z^3 + z^4 + \cdots)$; Africa, Europe, and Australia are restricted to no more than 5, so their generating functions are $(z + z^2 + z^3 + z^4 + z^5)$. Thus, the generating function for placement overall is their product:

$$(z + z^2 + z^3 + z^4 + \cdots)^2 (z + z^2 + z^3 + z^4 + z^5)^3$$

(c) **(4 points)** *Using algebraic manipulations of your generating function, determine its z^{25} coefficient.*

$$\begin{aligned}(z+z^2+\cdots)^2(z+z^2+\cdots+z^5)^3 &= z^5(1+z+z^2+\cdots)^2(1+z+\cdots+z^4)^3\\ &= z^5\left(\frac{1}{1-z}\right)^2\left(\frac{1-z^5}{1-z}\right)^3\\ &= z^5(1-z^5)^3\frac{1}{(1-z)^5}\\ &= z^5(1-3z^5+3z^{10}-z^{15})\sum_{n=0}^{\infty}\binom{n+4}{4}z^n\\ &= (z^5-3z^{10}+3z^{15}-z^{20})\sum_{n=0}^{\infty}\binom{n+4}{4}z^n\end{aligned}$$

We see four terms in this product that could contribute to a z^{25} term: $(z^5)\left[\binom{20+4}{4}z^{20}\right]$, $(-3z^{10})\left[\binom{15+4}{4}z^{15}\right]$, $(3z^{15})\left[\binom{10+4}{4}z^{10}\right]$, and $(z^{20})\left[\binom{5+4}{4}z^5\right]$. Adding them together, we get the term $\left[\binom{24}{4}-3\binom{19}{4}+3\binom{14}{4}-\binom{9}{4}\right]z^{20}$, so the coefficient of z^{20} is $\binom{24}{4}-3\binom{19}{4}+3\binom{14}{4}-\binom{9}{4}$. Note that this is the same as the answer to part (a), since it is counting the same thing.

5. **(12 points)** *A six-digit number is a string of six digits without a leading zero.*

   (a) **(4 points)** *Determine a generating function for a_n, the number of six-digit numbers with n even digits. The generating function need not be algebraically simplified.*

   The first digit could be any of 5 odd numbers, which will not increment our count, or any of 4 even numbers, which will increment our count of even numbers by one. Thus, a generating function for the selection of the first digit is $(5+4z)$. For the five subsequent digits, the selection process admits 5 odd numbers and 5 even ones, so their generating functions are $(5+5z)$. Multiplying the selection functions for all 6 digits,w e get $(5+4z)(5+5z)^5$.

   (b) **(4 points)** *How many six-digit numbers contain both an odd digit and an even digit?*

   We could actually use the generating function above to find this, if we wish: the expansion of the generating function is not difficult, and one could add together the z^1 through z^5 coefficients to get the result here.

   However, one can also use inclusion-exclusion. Let X consist of all six-digit numbers, so $|X| = 9\cdot 10^5 = 900000$. Then, let A consist of those numbers with no even digits, so $|A| = 5^6 = 15625$, and let B consist of those numbers with no odd digits, so $|B| = 4\cdot 5^5 = 12500$. Remarkably, $A\cap B$ is empty, since every number needs some digit, so our inclusion-exclusion expression is simply $9\cdot 10^5 - 5^6 - 4\cdot 5^5 = 871875$.

   (c) **(4 points)** *How many six-digit numbers contain at least one digit less than 2?*

   There are $9\cdot 10^5 = 900000$ six-digit numbers in total, and there are $8^6 = 262144$ without a 0 or 1 appearing as digits, so there are $9\cdot 10^5 - 8^6 = 637856$ six-digit numbers which *do* have a 0 or 1 appearing in their decimal representation.

6. **(12 points)** *For the following relations on the positive integers, determine (with an argument or counterexample) if they are reflexive, symmetric, and/or transitive. If they are equivalence relations, briefly describe the equivalence classes.*

(a) **(6 points)** *a is related to b if a is a multiple of b.*

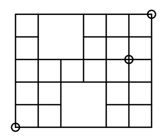
This is reflexive (since every number is a multiple of itself) and transitive (if $a = kb$ and $b = \ell c$, then $a = k(\ell c) = (k\ell)c$), but not symmetric (4 is a multiple of 2, but 2 is not a multiple of 4).

(b) **(6 points)** *a is related to b if $\lfloor \frac{a}{3} \rfloor = \lfloor \frac{b}{3} \rfloor$.*

This is reflexive, since $\lfloor \frac{a}{3} \rfloor = \lfloor \frac{b}{3} \rfloor$, and symmetric, by symmetry of equality (if $\lfloor \frac{a}{3} \rfloor = \lfloor \frac{b}{3} \rfloor$, then $\lfloor \frac{b}{3} \rfloor = \lfloor \frac{a}{3} \rfloor$). Transitivity likewise comes from the transitive property of equality: if $\lfloor \frac{a}{3} \rfloor = \lfloor \frac{b}{3} \rfloor$ and $\lfloor \frac{b}{3} \rfloor = \lfloor \frac{c}{3} \rfloor$, then $\lfloor \frac{a}{3} \rfloor = \lfloor \frac{c}{3} \rfloor$.

The equivalence classes are groups of numbers with the same result when they are divided by 3 and rounded down: so $\{1, 2\}$ is an equivalence class, as are $\{3, 4, 5\}$, $\{6, 7, 8\}$, and so forth.

7. **(12 points)** *Consider the following grid, in which the points $(3,1)$ and $(2,4)$ are missing.*



(a) **(9 points)** *How many ways are there to walk directly from the lower left corner of the grid to the upper right?*

Let X consist of all grid-walks from $(0,0)$ to $(6,5)$ on a grid without any missing points, and let A and B be those walks which pass through $(3,1)$ and $(2,4)$ respectively. Obviously, to find walks on the given grid, we want to avoid these two points, so walks on the grid with the omitted point are those in $\overline{A \cup B}$. Since we have a walk consisting of 6 steps to the east and 5 steps to the north, an element of X can be identified by selecting 6 of 11 steps as eastwards and leaving the rest as northwards, so $|X| = \binom{11}{6}$. Walks in A can be identified as a walk from $(0,0)$ to $(3,1)$, which can be done in any of $\binom{4}{3}$ ways, together with a walk from $(3,1)$ to $(6,5)$, which can be done in any of $\binom{7}{3}$ ways, so $|A| = \binom{4}{3}\binom{7}{3}$. Likewise, $|B| = \binom{6}{2}\binom{5}{4}$. Finally, we note that $A \cap B$ is empty, since no direct path visits both points. By inclusion-exclusion, we thus determine that the desired number of valid walks is

$$|\overline{A \cup B}| = \binom{11}{5} - \binom{4}{3}\binom{7}{3} - \binom{6}{2}\binom{5}{4} = 247$$

(b) **(3 points)** *How many ways are there to walk directly through the grid from the lower left to upper right if one* must *walk through the point $(5,3)$?*

Let X consist of all walks from $(0,0)$ to $(5,3)$, ignoring the possibility of missing points, and Y consist of all walks from $(5,3)$ to $(5,6)$; to account for the missing points, we consider the subsets A of X, consisting of paths from $(0,0)$ to $(5,3)$ through $(3,1)$. The number of paths from $(0,0)$ to $(5,3)$ *avoiding* $(3,1)$ is thus $|X| - |A|$, while the number of paths from $(5,3)$ to $(5,6)$ is simply $|Y|$. Since a path through $(5,3)$ is constructed by picking one of each of these subpaths and attaching them at $(5,3)$, the total number of paths from $(0,0)$ to $(5,6)$ passing through $(5,3)$ and avoiding the removed points is $(|X| - |A|)|Y|$. We can easily calculate $|X|$ to be $\binom{8}{3}$, $|A|$ to be $\binom{4}{1}\binom{4}{2}$, and $|Y|$ to be $\binom{3}{2}$, so our result is

$$\left[\binom{8}{3} - \binom{4}{1}\binom{4}{2}\right]\binom{3}{2} = 96$$

8. **(6 point bonus)** *Find the following enumerations:*

   (a) **(3 point bonus)** *The number of functions from* $\{1, 2, \ldots, 10\}$ *to* $\{a, b, c\}$ *such that each letter is the image of at least two numbers.*

   (b) **(3 point bonus)** *The number of equivalence relations on* $\{1, 2, 3, 4, 5\}$.

   An equivalence relation is uniquely determined by its equivalence classes, so the number of equivalence relations is exactly the number of partitions of this five-element set into 5 or fewer sets. This can be succinctly stated as the sum $S(5, 1) + S(5, 2) + S(5, 3) + S(5, 4) + S(5, 5)$ and can be derived from the formula for the Stirling numbers; other approaches also exist. This number is called the fifth *Bell number*, B_5.

> Die ganzen Zahlen hat der liebe Gott gemacht: alles anderere ist Menschenwerk. [God created the natural numbers: all else is the work of man.]
>
> —Leopold Kronecker