1. **(12 points)** *We play a game in which we roll a seven-sided die; we call a sequence of rolls "good" if it does not have two consecutive rolls of a number 3 or less (i.e. containing the subsequence 11, 12, 13 21, 22, 23, 31, 32, or 33).*

   (a) **(6 points)** *Find a recurrence relation and initial conditions for a_n, the number of good sequences of length n.*

   A good sequence of length n which ends in 4, 5, 6, or 7 can be constructed by preceding the last roll with any good sequence of length $n-1$; there are a_{n-1} ways to select a good sequence of length $n-1$, and 4 ways to select a suffix, so this accounts for $4a_{n-1}$ sequences.

   A good sequence of length n which ends in 1, 2, or 3, on the other hand, must be preceded by 4, 5, 6, or 7 to avoid two the prohibited pairs. Before these two digits, any good sequence of length $n-2$ will suffice. We thus have a construction built from three parts selectable in a_{n-2}, 4, and 3 ways, so this case accounts for $12a_{n-2}$.

   Since every possible ending for a sequence has been accounted for, these two cases describe all the good sequences of length n, so $a_n = 4a_{n-1} + 12a_{n-2}$.

   There is one sequence of length zero (the empty sequence) and 7 sequences of length one, so $a_0 = 1$ and $a_1 = 7$.

   (b) **(3 points)** *Find a closed-form (non-recurrence) formula for a_n.*

   The characteristic polynomial of $a_n = 4a_{n-1} + 12a_{n-2}$ is $x^2 - 4x - 12$, which has roots 6 and -2. Thus the general solution to this recurrence is $a_n = k_1(-2)^n + k_2 6^n$. Plugging in the initial conditions, we find that

$$\begin{cases} 1 = k_1 + k_2 \\ 7 = -2k_1 + 6k_2 \end{cases}$$

   which has solution $k_1 = -\frac{1}{8}$, $k_2 = \frac{9}{8}$. Thus, the closed form of a_n is

$$a_n = \frac{9(6^n) - (-2)^n}{8}$$

2. **(12 points)** *We have fifteen tasks to distribute to our four distinct employees. We must give each employee between 2 and 5 tasks.*

   (a) **(6 points)** *How many ways can we distribute the tasks if all the tasks are considered to be identical?*

   We may pre-emptively assign two tasks to each employee, leaving seven which need to be distributed to the four employees with no more than 3 assigned to each (since three, plus the two already assigned, is the upper limit of five tasks per employee). This is identical to seeking non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = 7$ in which each $x_i \leq 3$. Let X consist of all nonnegative solutions to $x_1 + x_2 + x_3 + x_4 = 7$, and let A_1, A_2, A_3, and A_4 consist of those solutions in which $x_1 > 3$, $x_2 > 3$, $x_3 > 3$, and $x_4 > 3$ respectively. The desired enumeration is the set in which none of these properties are satisfied, i.e. $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}|$. This may be calculated by inclusion-exclusion after determining $|X|$ and the sizes of all intersections of A_i. It is easy to show that $|X| = \binom{10}{3}$. We may calculate $|A_1|$ (and by symmetry, $|A_i|$ in general) by pre-emptively assiging 4 to x_1 and distributing the remaining 3 jobs $\binom{6}{3}$ ways. Furthermore, we may see that

$|A_i \cap A_j| = 0$, since violation of two properties would require assigning at least 4 to both of x_i and x_j, exceeding the total of 7 which we have to distribute. Thus,

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |X| - 4|A_i| = \binom{10}{3} - 4\binom{6}{3}$$

This problem may also be solved with generating functions, which involves calculating the z^{16} coefficient of

$$(z^2+z^3+z^4+z^5)^4 = z^8(1+z+z^2+z^3)^4 = z^8\frac{(1-z^4)^4}{(1-z)^4} = (z^8-4z^12+6z^16-4z^{20}+z^{24})\sum_{n=0}^{\infty}\binom{n+3}{3}z^n$$

(b) **(6 points)** *Describe a method for calculating the number of ways to distribute the tasks if all the tasks are considered to be distinct. You need not actually perform the calculation.*
This requires the unse of an exponential generating function, since the tasks are distinct (or ordered). Each employee is assigned between 2 and 5 tasks, so the task-assignment function for a single employee is

$$\frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}$$

and the generating function for assigning n distinct tasks under this restriction is thus

$$\left(\frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\right)^4$$

It now remains to pull the useful information from this generating function: the relevant piece of data is the coefficient of $\frac{z^{15}}{15!}$, since we wish to distribute 15 tasks. The relevant number, for the record, is 418738320, but this cannot be easily determined without computer-algebra systems.

3. **(12 points)**

(a) **(4 points)** Determine a generating function for the number of unordered partitions of n into parts of size 1, 2, or 5.
If we have x_1 partitions of size 1, x_2 partitions of size 2, and x_5 partitions of size 5, then partitions of n are described by solutions to

$$x_1 + 2x_2 + 5x_5 = n$$

Selection of x_1 allows us to contribute any integer number towards the total, so the selection function for x_1 is $(1+z+z^2+z^3+\cdots) = \frac{1}{1-z}$; selection of x_2 may contribute an even number towards the total, so the selection function for x_2 is $(1+z^2+z^4+z^6+\cdots) = \frac{1}{1-z^2}$; and selection of x_5 may contribute a multiple of 5 towards the total, so the selection function for x_5 is $(1+z^5+z^{10}+z^{15}+\cdots) = \frac{1}{1-z^5}$, so the generating function described by selecting each in turn is

$$(1+z+z^2+z^3+\cdots)(1+z^2+z^4+z^6+\cdots)(1+z^5+z^{10}+z^{15}+\cdots) = \frac{1}{(1-z)(1-z^2)(1-z^5)}$$

(b) **(4 points)** *Determine a recurrence relation and initial conditions for the number of* ordered *partitions of n into parts of size 1, 2, or 5. (e.g. $6 = 5 + 1$ and $6 = 1 + 5$ are considered distinct partitions of 6.*

Let a_n be the number of such ordered partitions. Consider the first term of a partition of n. If it is 1, then the remainder is a partition of $n-1$, which can be realized in a_{n-1} ways. If it is 2, the remainder is a partition of $n-2$, which can be realized in a_{n-1} ways; if it is 5, the remainder is a partition of $n-5$, which can be realized in a_{n-5} ways. Adding up these disparite cases, we see that $a_n = a_{n-1} + a_{n-2} + a_{n-5}$. Now, we unfortunately need initial conditions for a_0 through a_4 for this fifth-order recurrence relation. Simple enumeration shows that $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_4 = 5$.

(c) **(4 points)** *Determine a generating function for the number of unordered partitions of n into an even number of parts.*

This one was unexpectedly difficult; I expected a transposition to do the trick, but it turns out to be messy.

4. **(12 points)**

(a) **(6 points)** *Determine the coefficient of z^n in the power-series expansion of $(z^2+z^3)(z+z^2+z^3+z^4+z^5+\cdots)^3$.*

We expand the series as such:

$$\begin{aligned}
(z^2+z^3)(z+z^2+z^3+z^4+z^5+\cdots)^3 &= (z^2+z^3)z^3(1+z+z^2+z^3+\cdots)^3 \\
&= (z^5+z^6)\frac{1}{(1-z)^3} \\
&= (z^5+z^6)\sum_{n=0}^{\infty}\binom{n+2}{2}z^n \\
&= \sum_{n=0}^{\infty}\binom{n+2}{2}z^{n+5} + \sum_{n=0}^{\infty}\binom{n+2}{2}z^{n+6}
\end{aligned}$$

which under appropriate index-shifting becomes

$$\sum_{n=5}^{\infty}\binom{n-3}{2}z^n + \sum_{n=6}^{\infty}\binom{n-4}{2}z^n = z^5 + \sum_{n=6}^{\infty}\left[\binom{n-3}{2}+\binom{n-4}{2}\right]z^n$$

so the coefficient in question is $\binom{n-3}{2}+\binom{n-4}{2}$ for $n \geq 6$ (and 1 for $n = 5$, and 0 for $n < 5$).

(b) **(6 points)** *Using an exponential generating function, find the number of arrangements of 5 letters of the word TENNESSEE.*

The exponential generating function will describe arrangements of n letters, and we will pull out only the $\frac{n^5}{5!}$ coefficient to find our answer. "T" can be either present or absent, so it has associated exponential choice function $1+z$; there can be anywhere from zero to four "E"s, so it has exponential choice function $1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}$; there can be up to two "N"s and two "S"s, so both have choice functions $(1+z+\frac{z^2}{2}$, so our overall generating function is:

$$\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}\right)(1+z)\left(1+z+\frac{z^2}{2}\right)^2$$
$$=\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}\right)\left(1+3z+4z^2+3z^3+\frac{5}{4}z^4+\frac{z^5}{4}\right)$$

combining terms which form z^5 in this, we see that such terms add up as such: $\frac{1}{24}\cdot 3+\frac{1}{6}\cdot 4+\frac{1}{2}\cdot 3+1\cdot\frac{5}{4}+1\cdot\frac{1}{4}=\frac{91}{24}$, so the term in question is $\frac{91\cdot 5!}{24}\left(\frac{z^5}{5!}\right)$, with coefficient $\frac{91\cdot 120}{24}=455$.
Note: the actual exam problem will be less algebraically involved. This is what happens when I rush stuff to print.

5. **(12 points)**

(a) **(4 points)** *A number is* squarefree *if it is not divisible by any square number greater than 1. How many squarefree positive integers are there less than 100?*

If a number is divisible by a square, it is in particular divisible by the square of a prime number (e.g. numbers divisible by 6^2 are guaranteed to be divisible by 2^2 and 3^2). Consider all the primes less than 10: 2, 3, 5, and 7. Divisibility by 2^2, 3^2, 5^2, or 7^2 will prevent a number from being squarefree. Let A_2, A_3, A_5, and A_7 be sets of numbers with these properties, and $X=\{1,2,3,\ldots,100\}$ (we may include 100 if we like, since it will be excluded in the end anyways); then our sought number is $|\overline{A_2\cup A_3\cup A_5\cup A_7}|$. $|A_2|=25$, since there are 25 numbers from 1 to 100 divisible by 4; likewise $|A_3|=11$, $|A_5|=4$, and $|A_7|=1$. $|A_2\cap A_3|=2$ (36 and 72 are divisible by both 4 and 9), and $|A_2\cap A_5|=1$ (100 is divisible by 4 and 25), but all other intersections are empty. Thus, by inclusion-exclusion, $|\overline{A_2\cup A_3\cup A_5\cup A_7}|=|X|-|A_2|-|A_3|-|A_5|-|A_7|+|A_2\cap A_3|+|A_2\cap A_5|=100-25-11-4-1+1+2=62$.

(b) **(4 points)** *How many surjections are there from $A=\{a,b,c,d,e,f\}$ to $B=\{1,2,3\}$?*

Let X be the set of all functions from A to B: $|X|=3^6$. Now consider A_1, A_2, and A_3 as sets of functions which fail to be surejctions in various ways: failing to map anything to 1, 2, and 3 respectively. So, for instance, A_1 consists of functions from A to $\{2,3\}$, of which there are 2^6; likewise, $|A_2|=|A_3|=2^6$. Intersections of 2 violations, $A_i\cap A_j$ consist only of those functions mapping nothing to i or j, leaving only one possible image; thus $|A_i\cap A_j|=1^6$. Lastly, $A_1\cap A_2\cap A_3$ is empty, since at least one of the element sof B must be in the image of a function. Thus, using inclusion-exclusion, we see that the number of functions meeting none of these surjection-failure conditions is:

$$|\overline{A_1\cup A_2\cup A_3}|=|X|-\binom{3}{1}|A_i|+\binom{3}{2}|A_i\cap A_j|-\binom{3}{3}|A_1\cap A_2\cap A_3|=3^-3\cdot 2^6+6\cdot 1^6-0$$

(c) **(4 points)** How many 6-digit numbers have at least one of 1, 2, or 3 as a digit?

Let X consist of all six-digit numbers, and let A be the set of numbers with neither a 1, a 2, nor a 3. Then $\overline{A}$ consists of those numbers with at least one of these three digits. Clearly $|X|=9\cdot 10^5=90000$, and $|A|=6\cdot 7^5$, so the set of 6-digit numbers not lacking a 1, 2, or 3 is $|\overline{A}|=|A|-|X|=9\cdot 10^5-6\cdot 7^5$.