1. **(12 points)** *For the following problem, the alphabet used is the standard 26-letter English alphabet, and "vowels" refers only to the letters A, E, I, O, and U.*

   (a) **(3 points)** *How many strings of five letters contain at least one vowel?*

   There are 26^5 strings in total, since there are 26 different possible letters in each position. There are 21^5 strings which consist only of consonants, since there are 21 consonants. Thus, there are $26^5 - 21^5$ fove-letter strings which do not consist entirely of consonants; that is, strings which contain at least one vowel.

   (b) **(3 points)** *How many strings of five letters contain at least one A, at least one B, and at least one C?*

   Let X be the set of all five-letter strings; let A, B, and C be, respectively, the sets of strings with no As, no Bs, and no Cs. Thus, the question above asks to count the strings which do not appear in any of A, B, and C; which is to say, strings in $\overline{A \cup B \cup C}$. As seen above, $|X| = 26^5$; since A, B, and C consist of strings using a 25-letter pool each, $|A| = |B| = |C| = 25^5$; likewise, those sets which consist of strings avoiding 2 letters have a pool of 24 letters to choose from, so $|A \cap B| = |A \cap C| = |B \cap C| = 24^5$, and similarly $|A \cap B \cap C| = 23^5$. Using the principle of inclusion-exclusion:

   $$|\overline{A \cup B \cup C}| = 26^5 - 3 \cdot 25^5 + 3 \cdot 24^5 - 23^5$$

   An exhaustive casewise consideration is also possible but is prone to error through miscounts, overcounts, or omission of cases.

   (c) **(4 points)** *Determine a generating function for a_n, the number of seven-letter strings with n vowels. The generating function need not be algebraically simplified.*

   For each letter, we either have a consonant, in which case the choice of that letter does not contribute a vowel, or a vowel, in which case the contribution of that letter increases the vowel count by one. Thus, choice of a single letter can yield zero vowels in 21 ways, and one vowel in 5 ways, so the generating function describing the vowel count in the selection of a single letter is $(21 + 5z)$. Performing this action 7 times gives us a 7-letter string, so the generating function for 7-letter strings with the number of vowels recorded in the exponent of z is $(21 + 5z)^7$.

   (d) **(2 points)** *How many seven-letter strings have exactly 3 vowels?*

   In the generating function determined above, we want the coefficient of z^3, which is easily shown by the binomial theorem to be $\binom{7}{3}(21)^4(5)^3$.

2. **(12 points)** *A polynomial can be represented by a sequence of coefficients for computational purposes: we consider the polynomial $a_0+a_1x+a_2x^2+a_3x^3+\cdots a_nx^n$ as the sequence $(a_0, a_1, \ldots, a_n)$.*

   (a) **(9 points)** *Write an algorithm to multiply the two nth-degree polynomials in $\{a_i\}$ and $\{b_i\}$, and return the result in $\{c_i\}$, using only loops and simple arithmetic operations.*

Input: sequences $a_0, a_1, a_2, a_3, \ldots, a_n$ and $b_0, b_1, b_2, b_3, \ldots, b_n$
Output: sequences $c_0, c_1, c_2, c_3, \ldots, c_{2n}$

```
set c_0, c_1, c_2, ..., c_{2n} to 0;
set i to 0;
while i ≤ n do
|   set j to 0;
|   while j ≤ n do
|   |   set c_{i+j} to c_{i+j} + a_i b_j;
|   |   set j to j + 1;
|   set i to i + 1;
return determined values c_0, c_1, c_2, ..., c_{2n}
```

This works in much the same way as we multiply polynomials by hand. We start with the constant term of our multiplicand (setting $i = 0$, so we consider the term a_0), and multiply one-by-one by every term of the multiplier, recording the results to be added to the appropriate terms of our result (here we add them to the result progressively, while a traditional hand-multiplication would wait until all were determined to add up rows). Then, we repeat this process with the linear term of our multiplicand (setting $i = 0$, and thus looknig at the term $a_1 x$), and continuing in such fashion until we have multiplied each individual term of the multiplicand by the multiplier.

(b) **(3 points)** *Justify and state a good asymptotic bound in big-O notation on the number of steps taken by your algorithm.*

The innermost set of instructions (adding a product to c_{i+j}) takes constant time, or $O(1)$ time. However, this instruction is repeated when j is 0, j is 1, when j is 2, when j is 3, and so forth up to when j is n, so the time timen by performing this set of instructions $n + 1$ times is $(n + 1) \cdot O(1)$, or the linear time $O(n)$. Finally, since this loop is itself repeated over values of i ranging from 0 to n, the algorithm as a whole takes a number of steps given by $(n + 1) \cdot O(n)$; that is to say, performing a linear procedure $n + 1$ times, which takes $O(n^2)$ steps, also known as quadratic time.

3. **(12 points)** *Let G be a graph with n vertices and maximum degree Δ.*

(a) **(6 points)** *Prove that the vertices of G can be colored with $(\Delta + 1)$ colors without using the same color on two adjacent vertices.*

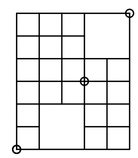
Capsule version: consider coloring the vertices one by one, in an arbitrary order. At each stage, we are coloring a vertex with $\leq \Delta$ neighbors, so there are $\leq \Delta$ colors adjacent to that vertex. Thus, there are $\leq \Delta$ colors which are used on vertices adjacent to our current vertex, which we are forbidden to use at that vertex; since there are $\Delta + 1$ colors in total, there must thus be at least one color which is not forbidden to use to color the vertex under consideration. Let us use such a color on the current vertex, and continue in such a fashion until all vertices are colored. Since we have shown that at each stage of the process we are capable of coloring the current vertex, this process must necessarily be successful in coloring all the vertices of G with $\Delta + 1$ colors.

(An inductive proof is also possible, for greater formalism)

(b) **(6 points)** *Prove that there is a set of at least $\frac{n}{\Delta+1}$ vertices in G such that no two of them are adjacent.*

Consider a coloring of the variety shown to be possible in part (a). This may be considered to be a partition of the vertices into "color classes", consisting of all vertices of color 1, all vertices of color 2, and so forth up to color $\Delta+1$. Since we are distributing n objects into $\Delta+1$ categories, the Pigeonhole Principle guarantees that one category has at least $\lceil\frac{n}{\Delta+1}\rceil$ objects in it – thus, there must be $\lceil\frac{n}{\Delta+1}\rceil$ vertices in this coloring which are all the same color, and by the conditions provided for our coloring, this set of vertices has no two adjacent to each other.

4. **(12 points)** *Consider the following grid, in which the points* $(2,1)$ *and* $(4,5)$ *are missing.*



(a) **(9 points)** *How many ways are there to walk directly from the lower left corner of the grid to the upper right?*

Let X consist of all grid-walks from $(0,0)$ to $(5,6)$ on a grid without any missing points, and let A and B be those walks which pass through $(2,1)$ and $(4,5)$ respectively. Obviously, to find walks on the given grid, we want to avoid these two points, so walks on the grid with the omitted point are those in $\overline{A\cup B}$. Since we have a walk consisting of 5 steps tot he east and 6 steps to the north, an element of X can be identified by selecting 5 of 11 steps as eastwards and leaving the rest as northwards, so $|X|=\binom{11}{5}$. Walks in A can be identified as a walk from $(0,0)$ to $(2,1)$, which can be done in any of $\binom{3}{2}$ ways, together with a walk from $(2,1)$ to $(5,6)$, which can be done in any of $\binom{8}{3}$ ways, so $|A|=\binom{3}{2}\binom{8}{3}$. Likewise, $|B|=\binom{9}{4}\binom{2}{1}$. Finally, we calculate $|A\cap B|$, which is the number of assemblies of *three* walks: from $(0,0)$ to $(2,1)$, which can be done in $\binom{3}{2}$ ways, from $(2,1)$ to $(4,5)$, which can be done in $\binom{6}{2}$ ways, and from $(4,5)$ to $(5,6)$, which can be done is $\binom{2}{1}$ways, so $|A\cap B|=\binom{3}{2}\binom{6}{2}\binom{2}{1}$. By inclusion-exclusion, we thus determine that the desired number of valid walks is

$$|\overline{A\cup B}|=\binom{11}{5}-\binom{3}{2}\binom{8}{3}-\binom{9}{4}\binom{2}{1}+\binom{3}{2}\binom{6}{2}\binom{2}{1}$$

(b) **(3 points)** *How many ways are there to walk directly through the grid from the lower left to upper right if one* must *walk through the point* $(3,3)$*?*

Let X consist of all walks from $(0,0)$ to $(3,3)$, ignoring the possibility of missing points, and Y consist of all walks from $(3,3)$ to $(5,6)$; to account for the missing points, we consider their subsets A, consisting of paths from $(0,0)$ to $(3,3)$ through $(2,1)$, and B, consisting of paths from $(3,3)$ to $(5,6)$ through $(4,5)$. The number of paths from $(0,0)$ to $(3,3)$ *avoiding* $(2,1)$ is thus $|X|-|A|$, and likewise the nubmer of paths from $(3,3)$ to $(5,6)$ avoiding $(4,5)$ is $|Y|-|B|$. Since a path through $(3,3)$ is constructed by picking one of each of these subpaths and attaching thenm at $(3,3)$, the total number of paths from $(0,0)$ to $(5,6)$ passing through $(3,3)$ and avoiding the removed points is $(|X|-|A|)(|Y|-|B|)$. We can easily calculate $|X|$ to be $\binom{6}{3}$, $|A|$ to be $\binom{3}{2}\binom{3}{1}$, $|Y|$ to be $\binom{5}{2}$, and $|B|$ to be $\binom{3}{1}\binom{2}{1}$, so our result is

$$\left[\binom{6}{3}-\binom{3}{2}\binom{3}{1}\right]\left[\binom{5}{2}-\binom{3}{1}\binom{2}{1}\right]$$

5. **(12 points)**

(a) **(7 points)** *Find a recurrence relation and initial conditions for the number of n-digit sequences of digits from $\{1, 2, 3, 4, 5, 6, 7\}$ with no two even numbers appearing consecutively.*

Let us call a sequence "good" if it does not have two adjacent even numbers, and denote by a_n the number of good sequences. The recurrence relation is motivated by the question: how do we build a good sequence from smaller good sequences? Appending an odd number to a good sequence clearly results in another good sequence; but appending an even number is only possible if an odd number precedes it. Thus, we can get every good sequence of length n in one of two ways: either append an odd number to a good sequence of length $n-1$, or append an odd number and then an even number to a good sequence of length $n-2$. There are 4 odd digits available to us, and a_{n-1} good sequences of length $n-1$, so the first construction method yields $4a_{n-1}$ good sequences of length n; the second construction method has an option of 4 different odd digits, 3 even digits, and a_{n-2} sequences of length $n-2$, so this construction can be completed in $12a_{n-2}$ different ways. Thus, there are $4a_{n-1} + 12a_{n-2}$ sequences of length n in total, so by our definition of a_n, this sequence is subject to the recurrence $a_n = 4a_{n-1} + 12a_{n-2}$.

For initial conditions, we consider the number of zero-digit good sequences (of which there is only 1, the null sequence), and the number of one-digit good sequences (of which there are 7, since any digit could be used). Thus $a_0 = 1$ and $a_1 = 7$.

(b) **(5 points)** *Solve the recurrence relation determined above.*

The characteristic equation of the recurrence relation above is $x^2 = 4x + 12$, which has roots $x = 6$ and $x = -2$, as can be determined by solving the quadratic. Thus, the general solution to the recurrence $a_n = 4a_{n-1} + 12a_{n-2}$ is $a_n = k_1 6^n + k_2(-2)^n$. The initial conditions are neceesary to determine k_1 and k_2. Looking at the particular cases of $n = 0$ and $n = 1$, the above equation becomes:

$$\begin{cases} 1 = k_1 + k_2 \\ 7 = 6k_1 - 2k_2 \end{cases}$$

Solving this system of equations gives $k_1 = \frac{9}{8}$ and $k_2 = -\frac{1}{8}$, so $a_n = \frac{9(6^n)-(-2)^n}{8}$.

6. **(12 points)**

(a) **(3 points)** *How many ways are there to rearrange the letters of the word "BARBER"?*

This is a rearragement of 2 identical items (the Bs), 2 identical items (the Rs), 1 unique item (the A) and another unique item (the E), so, since we want to position 4 classes of items, a multinomial coefficient is ideal, giving $\binom{6}{2,2,1,1} = 180$.

(b) **(4 points)** *How many rearrangements of the letters of the word "BARBER" do not have the "B"s adjacent to each other?*

We take the above-counted set, and subtract off those in which the Bs are adjacent. To count the arrangements with the Bs adjacent, we consider BB as a monolithic element in the rearrangement, so we now have 5 letters, 2 identical (the Rs), and 3 unique (the A, E, and BB). So there are $\binom{5}{2,1,1,1} = 60$ arrangements in which the Bs are adjacent to each other, so there are $\binom{6}{2,2,1,1} - \binom{5}{2,1,1,1} = 120$ arrangements in which they are not adjacent.

(c) **(5 points)** *Construct (but do not algebraically simplify) an exponential generating function for the number of n-letter strings which can be constructed from the letters of the word "BARBER".*

Let us consider this as a product of 4 selection functions: one choosing to place Bs, one placing Rs, one placing Es, and one placing As. Due to the multiplication principle of exponential generating functions, all rearrangements will be enumerated merely by performing multiplications, so we need to only write our base selection functions to select by the number of letters chosen. For instance, there are 3 possible words to be produced with B alone: the null word, the word "B", and the word "BB". Classifying these words by length, this corresponds to the generating function $(1)\frac{z^0}{0!} + (1)\frac{z^1}{1!} + (1)\frac{z^2}{2!} = 1 + z + \frac{z^2}{2}$. The selection function for words made from the letter R will be indentical. A and E, however only admit two possibilities: a zero-letter or a one-letter word, so their selection functions are $(1)\frac{z^0}{0!} + (1)\frac{z^1}{1!} = 1 + z$. Thus, multiplying all these together, we have

$$(1 + z + \frac{z^2}{2})^2(1 + z)^2$$

Note: actually multiplying these out is beyond the scope of this exam, however, if one were to do so, we could find the nubmer of one-letter, two-letter, etc. words constructable with these letters from the expansion and canonical representation of the exponential generating function:

$$1 + 4z + 14\left(\frac{z^2}{2!}\right) + 42\left(\frac{z^3}{3!}\right) + 102\left(\frac{z^4}{4!}\right) + 180\left(\frac{z^5}{5!}\right) + 180\left(\frac{z^6}{6!}\right)$$

Note, in particular, that the coefficient of $\frac{z^6}{6!}$ is, as would be expected, the answer to part (a).

7. **(12 points)** *Attila, Béla, Cili, and Desző are dividing up a collection of identical stamps. They have decided that none of them should get more than 8 or fewer than 2 stamps.*

   (a) **(4 points)** *Using classical methods, determine how many ways there are for them to distribute the stamps if there are 20 stamps total.*

   Let us pre-emptively hand out the requisite 2 stamps to each person, depleting our stock by 8; thus, this problem is identical to the question of how to give each person between zero and six stamps from a pool of 12.

   Let X consist of all possible distributions of stamps to the four philatelists; Let A_1, A_2, A_3, and A_4 represent those distributions which give a particular person more than 6 stamps.

   Using the distribution statistic for undistinguished objects to distinct recipients (or the "balls and walls" argument used to justify that statistic) we may see that $|X| = \binom{15}{3}$.

   Now, A_1 (and by analogy the other A_is) consists of those arrangements in which Atilla (or someone else, in the case of the other A_i) has at least 7 stamps. If we assign Attila a ration of 7 stamps, then there are 5 left to be distributed; this can be done in $\binom{8}{3}$ ways, so $|A_i| = \binom{8}{3}$.

   If we consider the intersection of some A_i and A_j, this would consist of those arrangements in which two people have at least 7 stamps each; distributing 12 stamps, this is a clear impossibility, so $|A_i \cap A_j| = 0$, and likewise larger intersections will be empty.

   Thus, by inclusion-exclusion,

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |X| - 4|A_i| = \binom{15}{3} - 4\binom{8}{3} = 231$$

.

(b) **(4 points)** *Construct a generating function for the number of ways to distribute n stamps in this manner. The generating function need not be algebraically simplified.*

Each person can get between 2 and 8 stamps, and each number of stamps can only be achieved in one way (since the stamps are all identical); thus, the generating function for distributing stamps to a single individual is $(z^2+z^3+z^4+z^5+z^6+z^7+z^8)$, and multiplying together each individual's stamp-selection finction, we get $(z^2+z^3+z^4+z^5+z^6+z^7+z^8)^4$.

(c) **(4 points)** *Using algebraic manipulations of your generating function, determine its z^{20} coefficient.*

$$\begin{aligned}
(z^2+z^3+z^4+z^5+z^6+z^7+z^8)^4 &= (z^2)^4(1+z+z^2+z^3+z^4+z^5+z^6)^4 \\
&= z^8\left(\frac{1-z^7}{1-z}\right)^4 \\
&= z^8(1-z^7)^4\frac{1}{(1-z)^4} \\
&= z^8(1-4z^7+6z^{14}-4z^{21}+z^{28})\sum_{n=0}^{\infty}\binom{n+3}{3}z^n \\
&= (z^8-4z^{15}+6z^{22}-4z^{29}+z^{36})\sum_{n=0}^{\infty}\binom{n+3}{3}z^n
\end{aligned}$$

We see two terms in this product that could contribute to a z^{20} term: $(z^8)\left[\binom{12+3}{3}z^{12}\right]$ and $(-4z^{15})\left[\binom{5+3}{3}z^5\right]$. Adding them together, we get the term $\left[\binom{15}{3}-4\binom{8}{3}\right]z^{20}$, so the coefficient of z^{20} is $\binom{15}{3}-4\binom{8}{3}=231$.

Note that this is the same as the answer to part (a), since it is counting the same thing.

8. **(12 points)** *For the following relations on the positive integers, determine (with an argument or counterexample) if they are reflexive, symmetric, and/or transitive. If they are equivalence relations, briefly describe the equivalence classes.*

(a) **(6 points)** *a is related to b if $|a-b|\leq 5$.*

Reflexivity can be shown easily: for all a, $|a-a|=0\leq 5$, so a relates to a.

This relation is also clearly symmetric: since $|a-b|=|b-a|$, it follows that $|a-b|\leq 5$ iff $|b-a|\leq 5$.

It is not, however, transitive: consider the coutnerexample $a=1$, $b=6$, and $c=11$. a is related to b since $|1-6|=5$ and b is related to c since $|6-11|=5$, but a is not related to c since $|1-11|=10\not\leq 5$.

(b) **(6 points)** *a is related to b if $a=2^n b$ for some (not necessarily positive) integer n.*

Reflexivity can be shown easily: for all a, $a=2^0 a$, so a relates to a.

This relation is also clearly symmetric: if a relates to b, then $a=2^n b$ for some n, but thus $b=2^{-n}a$, so b relates to a as well.

Transitivity is a little more difficult: given a relating to b, $a=2^n b$ for some n, and given b relating to c, $b=2^m c$ form some m (note: n and m need not be equal), so $a=2^n(2^m c)=2^{m+n}c$; thus a relates to c.

This is an equivalence relation, so it partitions the positive integers into equivalnce classes. The equivalence classes consist of numbers whose ratio is a power of 2: in particular, each equivalence class can be characterized by an odd number ℓ and consists of all numbers of the form $2^k\ell$ for $k \geq 0$.