For full credit show all of your work (legibly!), unless otherwise specified. This exam is closed-notes and calculators may not be used. Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponents, factorials, binomial coefficients, and multinomial coefficients.

1. **(12 points)** For the following problem, the alphabet used is the standard 26-letter English alphabet, and "vowels" refers only to the letters A, E, I, O, and U.

   (a) **(3 points)** How many strings of five letters contain at least one vowel?

   (b) **(3 points)** How many strings of five letters contain at least one A, at least one B, and at least one C?

   (c) **(4 points)** Determine a generating function for a_n, the number of seven-letter strings with n vowels. The generating function need not be algebraically simplified.

   (d) **(2 points)** How many seven-letter strings have exactly 3 vowels?

2. **(12 points)** A polynomial can be represented by a sequence of coefficients for computational purposes: we consider the polynomial $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n$ as the sequence $(a_0, a_1, \ldots, a_n)$.

   (a) **9 points** Write an algorithm to multiply the two nth-degree polynomials in $\{a_i\}$ and $\{b_i\}$, and return the result in $\{c_i\}$, using only loops and simple arithmetic operations.

   Input: sequences $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$
   Output: sequences $c_1, c_2, c_3, \ldots, c_{2n}$

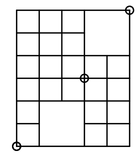
   (b) Justify and state a good asymptotic bound in big-O notation on the number of steps taken by your algorithm.

3. **(12 points)** Let G be a graph with n vertices and maximum degree Δ.

   (a) **(6 points)** Prove that the vertices of G can be colored with $(\Delta + 1)$ colors without using the same color on two adjacent vertices.

   (b) **(6 points)** Prove that there is a set of at least $\frac{n}{\Delta+1}$ vertices in G such that no two of them are adjacent.

4. **(12 points)** Consider the following grid, in which the points $(2,1)$ and $(4,5)$ are missing.



(a) **(9 points)** How many ways are there to walk directly from the lower left corner of the grid to the upper right?

(b) **(3 points)** How many ways are there to walk directly through the grid from the lower left to upper right if one *must* walk through the point $(3,3)$?

5. **(12 points)**

   (a) **(7 points)** Find a recurrence relation and initial conditions for the number of n-digit sequences of digits from $\{1, 2, 3, 4, 5, 6, 7\}$ with no two even numbers appearing consecutively.

   (b) **(5 points)** Solve the recurrence relation determined above.

6. **(12 points)**

(a) **(3 points)** How many ways are there to rearrange the letters of the word "BARBER"?

(b) **(4 points)** How many rearrangements of the letters of the word "BARBER" do not have the "B"s adjacent to each other?

(c) **(5 points)** Construct (but do not algebraically simplify) an algebraic generating function for the number of n-letter strings which can be constructed from the letters of the word "BARBER".

7. **(12 points)** Attila, Béla, Cili, and Desző are dividing up a collection of identical stamps. They have decided that none of them should get more than 8 or fewer than 2 stamps.

   (a) **(4 points)** Using classical methods, determint how many ways there are for them to distribute the stamps if there are 20 stamps total.

   (b) **(4 points)** Construct a generating function for the number of ways to distribute n stamps in this manner. The generating function need not be algebraically simplified.

   (c) **(4 points)** Using algebraic manipulations of your generating function, determine its z^{20} coefficient.

8. **(12 points)** For the following relations on the positive integers, determine (with an argument or counterexample) if they are reflexive, symmetric, and/or transitive. If they are equivalence relations, briefly describe the equivalence classes.

9. **(6 points)** a is related to b if $|a - b| \leq 5$.

10. **(6 points)** a is related to b if $a = 2^n b$ for some (not necessarily positive) integer n.