1.1.4. *This number trick involves your favorite three-digit number and your age. Multiply your age by 7. Subtract 1 from the result. Multiply by 11. Add 8. Multiply by 13. Add your favorite three-digit number. Subtract your age. Add 39. Show that the last three digits are always your original three-digit number and the first digits are your age.*

Let us call your age x and your favorite three-digit number y. The expected result described is $1000x + y$; let us show that this indeed results from the given process: applying each instruction in turn, we get the expression $[(x \cdot 7 - 1) \cdot 11 + 8] \cdot 13 + y - x + 39$. Simplifying this expression, we get $13(77x - 11 + 8) + y - x + 39 = 1001x - 39 + y - x + 39 = 1000x + y$, as hoped for.

1.1.6. *One version of the game Nim starts with two stacks of three coins. A player can remove any number of coins from one stack on their turn. The last player to remove a coin wins the game. What strategy should the second player use so that she always wins the game?*

Let us denote the situation where the stacks have k and ℓ coins in them in some order as an unordered pair (k, ℓ). So any player confronted with $(0, 1)$, $(0, 2)$, or $(0, 3)$ can win. No position forces the player confronted with it to produce the position $(0, 2)$ or $(0, 3)$, but any player confronted with $(1, 1)$ must produce $(0, 1)$, insuring the other player's victory. Thus, any player confronted with $(1, 2)$ or $(1, 3)$ can win, since they can present their opponent with the losing configuration $(1, 1)$. Since both $(0, 2)$ and $(1, 2)$ are victorious configurations, $(2, 2)$ is a losing position, since it forces one to hand their opponent victory. Thus, $(2, 3)$ is a winning position, since one can remove one coin to give one's opponent the losing position $(2, 2)$. Since the first player must give the second player one of $(0, 3)$, $(1, 3)$, or $(2, 3)$, the second player necessarily has a winning configuration. Noting that the losing positions discovered above are $(0, 0)$, $(1, 1)$, and $(2, 2)$, the second player's strategy is obvious: however many coins the first player removes at each stage, remove enough coins from the taller stack to make the stacks equal in height.

1.1.12. *Twenty-five students are seated in a square arrangement with 5 rows of 5 desks each. The teacher tells the students to switch desks so that each student is switched to a non-diagonally adjacent desk. Can all the students switch to a new desk simultaneously?*

If we color the squares in a checkerboard pattern, so that 13 are black and 12 are white, we see that black squares are adjacent only to white squares and vice-versa. Thus, every student currently in a black square must go to a white square and vice versa, so the 13 students currently at black-square desks must switch to the 12 desks in white-square spaces, which is clearly impossible to do without two going to the same desk.

1.1.16a. *A* $4 \times 4 \times 4$ *cube is to be cut into sixty-four* $1 \times 1 \times 1$ *cubes. What is the minimum number of cuts needed if intermediate arrangement of the resulting pieces is allowed?*

We want to get maximum value out of each cut, so we want to use each cut to divide up every single cube-piece on the table. Thus, with the first cut, we cut the cube into two halves; with the second, we cut both halves into four quarters, and so forth, so after n cuts we have 2^n pieces. So using each cut to its fullest potential, we would need at least 6 cuts to divide a cube into $2^6 = 64$ pieces. We can see by explicit construction that 6 cuts in fact suffice: cut in half lengthwise and rearrange the cube-halves to form a $2 \times 4 \times 8$ length, and cut lengthwise, so that after two cuts we have four $1 \times 4 \times 4$ plates. Stacking these plates, we repeat this process in the two other dimensions to get 64 $1 \times 1 \times 1$ cubes after 6 cuts.

1.1.16b. *An $n \times n \times n$ cube is to be cut into n^3 $1 \times 1 \times 1$ cubes. What is the minimum number of cuts needed if intermediate arrangement of the resulting pieces is allowed?*

We can get an easy lower bound on this: since we need at least k cuts to produce 2^k pieces, we need at least $\log_2(n^3)$ cuts to produce n^3 pieces. Thus we know that at least $\lceil \log_2(n^3) \rceil$ cuts are necessary. This isn't quite sufficient, though: we need 6 cuts to reduce a $3 \times 3 \times 3$ cube to . We can get a pretty good upper bound by noting that if we just cut dimension-by-dimension, we can reduce to $1 \times n \times n$ plates in $\lceil \log_2 n$ cuts, and repeating this in each dimension, we get an algorithm we can complete in $3\lceil \log_2 n \rceil$ cuts. This range is fairly small, and we can in fact verify that it's identically $3\lceil \log_2 n \rceil$ using more advanced techniques, namely a recursive formula.

1.2.6. *Show by induction that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.*

We start by observing the base case $n = 1$, which is simple: $\sum_{i=1}^{1} i^3 = 1^3 = 1 = \frac{1^2 2^2}{4}$. Now we proceed to an inductive step: we are given that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, and wish to show that $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$. We can do this with some algebraic manipulation:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\
&= \left(\frac{n^2(n+1)^2}{4}\right) + (n+1)(n+1)^2 \\
&= \left(\frac{n^2}{4} + (n+1)\right)(n+1)^2 \\
&= \left(\frac{n^2+4n+4}{4}\right)(n+1)^2 \\
&= \left(\frac{(n+2)^2}{4}\right)(n+1)^2 = \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}$$

1.2.10. *Find a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ and prove that it holds for all $n \geq 1$.*

Calculating this for small n, we get $\frac{1}{2}$, $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, and $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$, so we hazard a guess that the sum is always $\frac{n}{n+1}$. We have shown already that this is true for $n = 1$, so we need only perform the inductive step. Armed with the knowledge that

$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$, this becomes easy:

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \left(\sum_{i=1}^{n} \frac{1}{i(i+1)}\right) + \frac{1}{(n+1)(n+2)} \\ &= \left(\frac{n}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= 1 - \frac{1}{n+2} = \frac{n+1}{n+2}\end{aligned}$$

1.2.20. *For any positive integer n, show that $2^{2n} - 1$ is divisible by 3.*

Let $a_n = 2^{2n} - 1$. We can observe that $a_1 = 3$ for the base case; for an inductive step, we assume a_n is divisible by 3 and attempt to show that a_{n+1} is. Observe that $a_{n+1} = 2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 4 \cdot (a_n + 1) - 1 = 4 \cdot a_n + 3$. Assuming a_n is divisible by 3, so is $4a_n$, and so thus is $4a_n + 3$, so a_{n+1} is also divisible by 3.

1.3.2. *In a combinatorics class of 50 students, 32 are male, 41 are right-handed, and 26 are right-handed males. How many left-handed female students are in the class?*

Let X be our universe of students; A be the set of male students, and B the set of right-handed students. We are told $|X| = 50$, $|A| = 32$, $|B| = 41$, and $|A \cap B| = 26$. The set of left-handed females is the set of all people who are neither right-handed nor male; that is, the complement of $A \cup B$. Thus the enumeration requested is $|\overline{A \cup B}|$. Using counting rules,

$$\begin{aligned}|\overline{A \cup B}| &= |X| - |A \cup B| \\ &= |X| - (|A| + |B| - |A \cap B|) \\ &= 50 - (32 + 41 - 26) = 3\end{aligned}$$

1.3.10. *Determine which of the properties of relations are satisfied by the following relations on positive integers i and j.*

1. *$i + j$ is even.* This is reflexive, since $i + i$ is always even. It is symmetric, since $i + j$ and $j + i$ have the same parity. And lastly, it is transitive: if $i + j$ is even, and $j + k$ is even, we may add those two expressions to note that $i + 2j + k$ is even; subtracting off the known-even $2j$ from this expression, we see that it follows that $i + k$ is even. This relation is thus an equivalence relation (and is in fact the same relation as congruence modulo 2).
2. *$i + j$ is odd.* This is not reflexive: $i + i$ is never odd. It is symmetric, since $i + j$ and $j + i$ have the same parity. It is nontransitive, as we can see by an example: $1 + 2$ is odd, $2 + 3$ is odd, but $1 + 3$ is even. This relation is in fact as nontransitive as possible: if $i + j$ is odd and $j + k$ is odd, $i + k$ will always be even.

3. $|i - j| \leq 10$. This is reflexive, since $|i - i| = 0 \leq 10$. It is symmetric, since $|i-j| = |j-i|$. It is not transitive, as can be exhibited by example: $|10-0| \leq 10$, and $|20 - 10| \leq 10$, but $|20 - 0| \not\leq 10$.
4. $i \geq j$. This is reflexive, since $i \geq i$. It is not symmetric, since if $i \geq j$, it does not follow that $j \geq i$ ($i = 2$ and $j = 1$ is an example). It is transitive, since if $i \geq j$ and $j \geq k$, it follows that $i \geq k$.

1.3.12. *A relation R on a set A is called circular if aRb and bRc implies that cRa, Show that if R is reflexive and circular, it is an equivalence relation.*

We need to show that symmetry and transitivity follow from reflexivity and circularity. Symmetry we show by assuming that some aRb, and proving that it follows that bRa. Rexflexivity gives us that bRb, so since aRb and bRb, it follows from circularity that bRa. To prove transitivity is simple given circularity and the just-proven symmetry: given aRb and bRc, circularity shows that cRa, and therefrom we may derive by symmetry that aRc.

1.3.18. *If $A \subseteq B$ and $C \subseteq D$, prove that*

1. $A \cap C \subseteq B \cap D$.
   Let $x \in A \cap C$. We shall show that it follows that $x \in B \cap D$. For x to be in $A \cap C$, it must be the case that $x \in A$ and $x \in C$. Since A is a subset of B, every element of A is in B, so $x \in B$. Likewise, since $x \in C$ and $C \subseteq D$, $x \in D$. Since $x \in B$ and $x \in D$, $x \in B \cap D$. We have shown that an arbitrarily chosen element of $A \cap C$ is in $B \cap D$, so $A \cap C \subseteq B \cap D$.
2. $A \cup C \subseteq B \cup D$.
   Let $x \in A \cup C$. We shall show that it follows that $x \in B \cup D$. For x to be in $A \cup C$, it must be the case that $x \in A$ or $x \in C$. Since A is a subset of B, every element of A is in B, so if $x \in A$, then $x \in B$. Likewise, since $C \subseteq D$, if $x \in C$, then $x \in D$. Since one of the premises $x \in A$ or $x \in C$ must be true, it follows that $x \in B$ or $x \in D$, so $x \in B \cup D$. We have shown that an arbitrarily chosen element of $A \cup C$ is in $B \cup D$, so $A \cup C \subseteq B \cup D$.

1.4.4. *Show that among any 13 integers, not necessarily consecutive, there are at least two whose difference is a multiple of 12.*

Consider a sorting function placing associating each of the twelve modular congruence classes modulo 12 with a signle bin, so that numbers are placed in one of twelve bins depending on their congruence class. Given 13 numbers $a_1, \ldots, a_{13}$, if we subject them to this sorting function, then by the pigeonhole principle two of the numbers a_i and a_j are in the same bin, and thus are congruent to each other; thus by the definition of modular congruence, $a_i - a_j$ is a multiple of 12.

1.4.8. *A basketball team plays 30 games in 20 days, playing at least one game every day.*

1. *Show that there must be a period of consecutive days during which the team plays exactly 9 games.*

Let a_i count the number of games played on the first i days. Since at least one game is played each day, $a_{i+1} > a_i$, so we have an increasing sequence

$$0 = a_0 < a_1 < a_2 < \cdots < a_{19} < a_{20} = 30$$

so we have 21 distinct numbers from 0 to 30. Any sequence of consecutive days we can get by subtracting these numbers from each other, so the number of games played from the ith day to the jth is $a_j - a_{i-1}$. Thus, we want to show that there are some elements of this sequence differing by 9. To this end, we craft a system of bins in which elements differ by 9, creating 20 bins with the following sorting scheme for the numbers zero through 30: $\{0, 9\}$, $\{1, 10\}$, …, $\{8, 17\}$, $\{18, 27\}$, $\{19, 28\}$, $\{20, 29\}$, $\{21, 30\}$, $\{22\}$, $\{23\}$, …, $\{28\}$. Since there are 21 values of $a_0, \ldots, a_{20}$ and 20 bins, two of them share a bin and thus have a difference of 9.

2. *Is there necessarily a period of consecutive days when exactly 10 games are played?*

   We proceed as above, given 21 numbers between 0 and 30, and try to devise a system of bins in which each difference is 10, with $\{0, 10\}$, $\{1, 11\}$, …, $\{9, 19\}$, $\{20, 30\}$, $\{21\}$, $\{22\}$, …, $\{29\}$. As above, sorting by this criterion gives us two nubmers in the same bin by the pigeonhole principle, and guarantees a difference of 10 by the construction of our sorting function.

1.4.10. *Show that among $n + 1$ different positive integers less than or equal to $2n$, there are always two that are relatively prime.*

Two consecutive numbers are always guaranteed to be relatively prime, so we can devise a system of n bins with intended occupants $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, …, $\{2n - 1, 2n\}$. Given $n + 1$ integers between 1 and $2n$, distributing them according to the above rule guarantees two in one bin, so there are two relatively prime numbers among our $n + 1$ chosen numbers. We may in fact make the stronger statement that among these $n + 1$ numbers, two of them are not only relatively prime, but consecutive.

1.4.17. *Prove that any subset of 53 numbers chosen from the set $\{1, 2, \ldots, 100\}$ must contain two numbers that differ by exactly 12, but need not contain a pair differing by 11.*

For the first part of this problem, a pigeonhole application requires that we partition $\{1, 2, \ldots, 100\}$ up into 52 or fewer bins so that each bin consists of numbers only differing by exactly 12 — each bin will be set up to receive either the pair $\{x, x + 12\}$ or the singleton $\{x\}$. We can in fact do this with a greedy method: start by building the 12 bins $\{1, 13\}$ through $\{12, 24\}$. We then repeat from $\{25, 37\}$ through $\{36, 48\}$, again from $\{49, 61\}$ through $\{60, 72\}$, and again from $\{73, 85\}$ to $\{84, 96\}$, and finally the singletons $\{97\}$, $\{98\}$, $\{99\}$, and $\{100\}$, for a grand total of 52 bins.

For a difference of 11, we come across a problem; this construction would give us 44 bins containing numbers from 1 to 88, one more bin for $\{89, 100\}$, and then the remaining 10 numbers would need singleton bins, for a total of 55 bins, entirely too many for a pigeonhole problem. We can in fact, by taking the least element of each bin, construct a counterexample set with 55 elements and no difference-11 pairs:

$$\{1, 2, \ldots, 11, 23, 24, \ldots, 33, 45, 46, \ldots, 55, 67, 68, \ldots, 77, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$$