2.1.10. *On the menu of a Chinese restaurant there are 7 chicken dishes, 6 beef dishes, 6 pork dishes, 8 seafood dishes, and 9 vegetable dishes.*

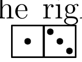
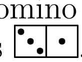
(a) *In how many ways can a family order if they choose exactly one dish of each kind?*

We may consider this a sequential problem of choosing a chicken dish, then a beef dish, and so forth. There are 7 options for the first choice, 6 for the second, 6 for the third, 8 for the fourth, and 9 for the fifth, making a total of $7 \cdot 6 \cdot 6 \cdot 8 \cdot 9 = 18144$ possible orders.

(b) *In how many ways can a family order if at most one dish of each kind is ordered?*

We may consider this a sequential problem as above, but with one extra option in each case: the family may elect for each of the seven chicken dishes or to order no chicken at all, for eight possible choices. Extending this to each decision, we see that there are 8 options for the first choice, 7 for the second, 7 for the third, 9 for the fourth, and 10 for the fifth, making a total of $8 \cdot 7 \cdot 7 \cdot 9 \cdot 10 = 35280$ possible orders (which includes some patent absurdities, such as opting not to order anything at all, but we have not been given a mandate to specifically exclude any cases; if you regard that as "not an order", however, you may subtract one from the number given above to get 35279 possible orders).

2.1.14. *A domino is a 1×2 rectangular tile divided into two square halves. The squares in a standard set of dominos are each marked with zero to six spots. If every domino is different, how many dominos are there in a complete set of dominos? In particular, explain why your answer is less than 49.*

The naïve solution is to consider the 7 possibilities (zero through six) on the left half; and then the 7 possibilities on the right half of a domino to get $7 \times 7 = 49$. This, however, overcounts: the domino  is the same as . To solve this more effectively we need to consider the distinct possibilities of a domino with the same number of spots on each half, which can happen in 7 ways, and a domino with different numbers on each half, which can occur in $7 \cdot 6 = 42$ ways. The latter case is overcounted, since the choice (i, j) is the same as the choice (j, i), so these 42 choices really describe only 21 distinct dominoes. Thus, the total number of possible dominoes is $21 + 7 = 28$.

2.1.18. *An $m \times n$ matrix is a rectangular array of numbers containing m rows and n columns. If every number in the matrix is either a 0 or a 1:*

(a) *How many $m \times n$ matrices are there?*

An $m \times n$ matrix has mn entries. For each of these entries, we have a choice to set it to 0 or to 1. Thus, a matrix is the result of mn decisions, each of which has 2 possible choices. We thus multiply mn instances of 2 together to get 2^{mn}.

(b) *How many $m \times n$ matrices are there with a single 1 in each row?*

Since each row contains exactly one "1", a row is dictated by the decision of position to place the 1 in, and all other positions are filled with zeroes. For such a decision, we have n possibilities, and each row's decision has no influence on the decision process in other rows, so that building a matrix is the process of making such a decision for each of the n rows. Thus, the number of matrices we can build under this constraint is the product of n instances of m, which is m^n.

2.1.20. *Let A be a set of n distinct elements. There is a one-to-one correspondence between binary relations on the set A and subsets $R \subseteq A \times A$.*

(a) *Compute the number of binary relations on A.* We know that $|A \times A| = n^2$; we also know, from one of the original bijections we crafted, that if a set S has k elements, there are 2^k possible subsets of S. There are thus $2^{\left(n^2\right)}$ possible subsets of $A \times A$, which are established above to be in a one-to-one correspondence with the relations on A, so there are $2^{\left(n^2\right)}$ relations on A.

(b) *A binary relation is symmetric if for every (a, b) in R, (b, a) is also in R. Compute the number of symmetric binary relations on A.*

In the previous problem we essentially answered each of the n^2 questions "is (a, b) in R?" for various values of a and b in either of two ways to get $2^{\left(n^2\right)}$. In this case we are again answering a number of independent questions to determine our relation, but the questions are of the form "are both of (a, b) and (b, a) in R, or are neither?" Again, we have 2 answers to each questions, but the questions each determine the membership of two pairs rather than one in R. However, we must be mindful of the case where $a = b$. The $a = b$ case induces n questions of the above form. When $a \neq b$, we have $n(n-1)$ choices of (a, b), but since two ordered pairs are mentioned in each question, this raises only $\frac{n(n-1)}{2}$ questions. We thus characterize a relation by answering $n + \frac{n(n-1)}{2}$ yes-no questions, which we can do in $2^{n+\frac{n(n-1)}{2}}$ different ways.

2.1.20c. *A binary relation is symmetric if for every (a, b) in R with $a \neq b$, (b, a) is not in R. Compute the number of symmetric binary relations on A.* ' Each of the relations (a, a) is either in R or not; thus we have 2^n ways to choose how the elements of A are self-related. For each unordered pair of distinct nubmers (a, b), however, we have 3 evident choices: (a, b) is in R while (b, a) is not; (b, a) is in R while (a, b) is not; or neither (a, b) nor (b, a) is in R. thus, for each unordered pair (of which there are $\frac{n(n-1)}{2}$, as detailed in the previous part of the questions) we have 3 choices of relation membership, so there are $3^{\frac{n(n-1)}{2}}$ ways to choose relation-membership for distinct elements of A. Thus, there are $2^n 3^{\frac{n(n-1)}{2}}$ antisymmetric relations which can be chosen.

2.1.23. *Find the smallest positive integer with exactly 18 positive divisors.*

We know, from an example in class (or enumerating the possible prime factorizations for a divisor) that if N has prime factorization $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then N has $(a_1+1)(a_2+1) \cdots (a_k+1)$ factors. For N to have 18 factors, then, the exponents in the prime factorization must be subject to $(a_1+1)(a_2+1) \cdots (a_k+1) = 18$. Since each $a_i > 1$, we seek to enumerate the possible ways 18 could be decomposed into a product of one or more numbers. There are not too many ways to do so, so they can be explicitly listed:

$$\begin{aligned} 18 &= 18 &&= (17+1) \\ 18 &= 9 \cdot 2 &&= (8+1)(1+1) \\ 18 &= 6 \cdot 3 &&= (5+1)(2+1) \\ 18 &= 3 \cdot 3 \cdot 2 &&= (2+1)(2+1)(1+1) \end{aligned}$$

so we have four possibilities: $N = p_1^{17}$, $N = p_1^8 p_2^1$, $N = p_1^5 p_2^2$, and $N = p_1^2 p_2^2 p_3^1$. To make N as small as possible, we choose the smallest primes possible for $p_1, \ldots, p_n$, and associate the smallest primes with the largest exponents to minimize the value. Thus, our four possibilities become $2^{17} = 131072$, $2^8 3^1 = 768$, $2^5 3^2 = 288$, and $2^2 3^2 5^1 = 180$, so 180 is the smallest number with 18 divisors.

2.2.4. *In a computer science department there are three graduate students and ten professors.*

(a) *In how many ways can each student be assigned an advisor if no professor advises all three students?*

Each student can be assigned to one of ten different advisors; thus, there are 10^3 assignments without constraint. However, there are 10 ways to assign all three to the same advisor, so there are $10^3 - 10 = 990$ assignments which do not assign them all to the same advisor.

(b) *In how many ways can each student be assigned a different advisor?*

The first student has a choice of ten possible advisors; the second, regardless of the first's choice, has exactly nine choices (whichever nine were unchosen), and the third has eight. Thus there are $10 \cdot 9 \cdot 8 = 720$ such arrangements.

2.2.8. *A single die is rolled five times in a rown.*

(a) *How many outcomes will result in five different numbers?*

The first roll can come up any of six ways. The second must come up with one of the five numbers which did not show up on the first roll, the third with one of the four remaining numbers, and so forth, so that the total number of possible outcomes is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.

(b) *How many outcomes will have the fifth number equal to an earlier number?*

It is easier to solve the complementary problem: how many outcomes have the fifth number distinct from all the previous numbers? To determine this set of outcomes, we choose the fifth roll any of 6 ways, and then the first four rolls can take on any value except the one already chosen, so that there are $6 \cdot 5^4$ possible such outcomes. There are 6^5 outcomes total, since each roll can have any of 6 results, so the number of outcomes in which the fifth number is equal to an earlier number is $6^5 - 6 \cdot 5^4 = 4026$.

2.2.13. (a) *How many strings of length 8 that contain exactly two vowels can be constructed from the English alphabet?*

We may select a string of 6 consonants in 21^6 ways, and choose a pair of vowels in 5^2 ways. However, the distribution of consonants and vowels in the word has not yet been chosen (and, for instance, AEBCDFGH and BACDEFGH are distinct words). We thus must choose 2 slots from the 8 given for vowels. This is $\binom{8}{2}$ in the parlance of Section 2.3, but in this section we may devise it by taking the $8 \cdot 7$ possible ordered selections of two slots, and dividing by 2. We thus have $(21^6)(5^2)\frac{8\cdot7}{2} = 60036284700$.

(b) *Repeat part (a) if the two vowels cannot be adjacent.*

Of our $\frac{8\cdot7}{2} = 28$ selected vowel-placements, 7 have the vowels adjacent: the first and second slots, the second and third slots, and so forth up to the seventh and eighth slots. Thus, we must exclude these placements, yielding $(21^6)(5^2)\left(\frac{8\cdot7}{2} - 7\right) = 45027213525$.

2.2.16. *How many five-digit numbers with distinct digits*

(a) *Contain at least one 0?*

There are $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216$ five-digit numbers with distinct digits total. Of these, $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ contain no zero, so there are $27216 - 15120 = 12096$ numbers which contain at least one 0.

(b) *Contain at least one 9?* There are $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216$ five-digit numbers total. Of these, $8 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 13440$ contain no nine, so there are $27216 - 13440 = 13776$ numbers which contain at least one 9.

(c) *Contain at least one 0 and at least one 9?*

Let A_0 be the set of numbers lacking a zero and A_9 the set of numbers lacking a nine, and let the universe X consist of all five-digit numbers with distinct digits. As observed above, $|A_0| = 15120$, $|A_9| = 13440$, and $|X| = 27216$. We seek numbers which have both a 0 and a 9: that is, those which are a member of neither A_0 nor A_9. Alternatively, non-membership in both A_0 and A_9 may be phrased as non-membership in $A_0 \cup A_9$, or membership in $\overline{A_0 \cup A_9}$. We know that

$$\begin{aligned}
|\overline{A_0 \cup A_9}| &= |X| - |A_0 \cup A_9| \\
&= |X| - (|A_0| + |A_9| - |A_0 \cap A_9|) \\
&= 27216 - (15120 + 13776 - |A_0 \cap A_9|) \\
&= |A_0 \cap A_9| - 1344
\end{aligned}$$

Now, we must find $|A_0 \cap A_9|$. This set, by the description of A_0 and A_9, consists of those numbers which have neither 0 nor 9 appearing in them. We thus have only 8 choices for the first digit, 7 for the second, and so forth, so $|A_0 \cap A_9| = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$, and our original calculation is $6720 - 1344 = 5376$.

Note: all of the above questions can also be answered with direct enumeration.

2.2.18. (a) *How many positive integers are divisors of both* 10^{40} *and* 20^{30}*?*

Note that $10^{40} = (2 \cdot 5)^{40} = 2^{40}5^{40}$ and $20^{30} = (2^2 \cdot 5)^{30} = 2^{60}5^{30}$. Thus, both have divisors of the form $d = 2^a3^b$. For d to be a divisor of 10^{40}, $0 \leq a \leq 40$ and $0 \leq b \leq 40$; for d to be a divisor of 20^{30}, $0 \leq a \leq 60$ and $0 \leq b \leq 30$. Thus, to be a divisor of both, it must be the case that $0 \leq a \leq 40$ and $0 \leq b \leq 30$. This allows 41 choices for a, and 31 for b, so there are $41 \cdot 31 = 1271$ shared factors.

(b) *How many positive integers are divisors of at least one of the integers* 10^{40} *and* 20^{30}*?*

Let A be the set of divisors of 10^{40} and B be the set of divisors of 20^{30}. Using the prime factorizations determined above, it is easy to see that $|A| = (40+1)(40+1) = 1681$ and that $|B| = (60+1)(30+1) = 1891$. We wish to find the number of integers which are divisors of either of 10^{40} or 20^{30}; that is, we want to find $|A \cup B|$. The solution to the previous problem was the number of integers in both A and B, so $|A \cap B| = 1271$. Thus, $|A \cup B| = |A| + |B| - |A \cap B| = 1681 + 1891 - 1271 = 2301$.