2.3.4. (a) *How many teams of 5 players can be chosen from a group of 10 players?*

Assuming team members have indistinugishable roles, this is a matter of selecting 5 distinct elements in no order from a set of 10; we can count the ways ot do this with the combination statistic $\binom{10}{5} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{120} = 252$.

(b) *How many teams will include the best player and exclude the worst player?*

To select a team with these properties, we start by selecting the best player, which is required. At this point we must still choose 4 players, and we have 8 possible players remaining to fill the team (we exclude the best player, who has already been chosen, and the worst player, whom we are forbidden to choose). Thus, we can fill the rest of the team in $\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{24} = 70$ different ways.

2.3.10. *Suppose that a club has eight male and nine female members. In how many ways can a four-person committee be chosen if*

(a) *Exactly two women committee members are to be chosen?*

We may consider the construction of a committee as a two-step process: select the women serving on it, and then the men. There are nine women, so there are $\binom{9}{2}$ ways to select two of them to serve on the committee;. likewise, there are $\binom{8}{2}$ ways to select two men to serve. Since the formation of a committee results from the independent performance of these two steps, we may form any of $\binom{9}{2}\binom{8}{2} = \frac{9 \cdot 8}{2}\frac{8 \cdot 7}{2} = 1008$ different committees.

(b) *At least two women committee members are to be chosen?*

We divide this up into several cases; either there are two women, three women, or four women on the committee. We see, from the argument in the previous section, that the first case contributes $\binom{9}{2}\binom{8}{2}$ possible committees; the second and third cases can be shown, via very simple modifications of the above argument, to contribute $\binom{9}{3}\binom{8}{1}$ and $\binom{9}{4}\binom{8}{0}$ possible committees. Thus, the total number of possible committees is

$$\binom{9}{2}\binom{8}{2} + \binom{9}{3}\binom{8}{1} + \binom{9}{4}\binom{8}{0} = 36 \cdot 28 + 84 \cdot 8 + 126 \cdot 1 = 1806$$

This result could also be obtained by looking at the number of committees total, which is $\binom{9+8}{4} = 2380$, and subtratcting off the number of male-dominated committees, which based on a casewise analysis similar to that above is $\binom{9}{1}\binom{8}{3} + \binom{9}{0}\binom{8}{4} = 504 + 70 = 574$.

2.3.14. *In a group of 30 ball bearings, 5 are defective. If 10 ball-bearings are chosen, what is the probability that none of them is defective?*

The number of ways to choose 10 ball-bearings from a pool of 30 is $\binom{30}{10}$, assuming the selection process does not distinguish among its ten selectees. The number of ways to choose 10 non-defective ball-bearings is simply the number of ways to choose 10 ball-bearings from the diminished pool of non-defective ball-bearings – that is, a pool of 25 possible choices. Thus, there are $\binom{25}{10}$ ways to choose 10 non-defective ball-bearings. The probability of such a selection is then:

$$\frac{\binom{25}{10}}{\binom{30}{10}} = \frac{\frac{25!}{10!15!}}{\frac{30!}{10!20!}} = \frac{25!20!}{30!15!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26} \approx 11\%$$

2.3.22. *In how many ways can eight identical rooks be placed on an ordinary* 8×8 *chessboard so that no two are in the same row or column? In how many ways, if each rook has a different color?*

We must place exactly one rook on each row, since we cannot place two on the same row and must place 8 among 8 rows. Thus, for each row, we must choose which column to put the rook on. The rook on the first row can be in any of 8 columns; the rook on the second row can go on any of the remaining seven, and so forth, for $8! = 5040$ placements total. If each rook also has a color, then we have any of eight colors for the rook in the first row, any of seven for the rook in the second row, etc., for $8!8! = 25401600$ possible placements total.

2.3.26. *In one version of the game of poker, seven cards are dealt to a player (the order of distribution is unimportant). How many different poker hands have three different pairs (two cards of the same rank) and one card of a fourth rank.*

We first want to select 3 distinct ranks which have no order among them (i.e. pairs of kings, sixes, and twos is the same as pairs of sixes, twos, and kings). There are $\binom{13}{3}$ ways to do this. Then, for each of the ranks we assign suits. This task does have an order, since having kings of hearts and clubs is a different hand than sixes of hearts and clubs, for instance. Thus, for each of the ranks (if we like, we can think of this as starting with the highest rank and working down) we select 2 suits; we can do this in $\binom{4}{2}$ ways for each rank, and there are thus $\binom{4}{2}^3$ suit assignments performed on these three ranks. Finally, we want one card not from any of these three ranks; this card can have any of 10 ranks and any of 4 suits, so it can be any of 40 different cards. Thus, the number of such hands is

$$\binom{13}{3}\binom{4}{2}^3 \cdot 40 = 2471040$$

2.4.4. (a) *Find the number of nonnegative integer solutions of* $x_1 + x_2 + x_3 + x_4 + x_5 = 24$.

This can be thought of as a distribution problem in which 24 indistinguishable objects are distributed among 5 recipients. One way to do this would be to place 4 "walls" among 24 "items", so that the walls separate the items into the portion for each recipient. Since there would be $24 + 4 = 28$ "slots" total, each of which is occupied by a wall or an item, and 4 wall-positions must be selected, there are $\binom{28}{4} = 20475$ soloutions.

(b) *What if each* x_i *must be at least 2?*

It would be useful to convert this to a problem already solved, and we may indeed do so by letting $x_i = y_i + 2$; then $x_i \geq 2$ if and only if y_i is a nonnegative integer.

Solutions to $x_1+x_2+x_3+x_4+x_5=24$ are simply solutions to $(y_1+2)+(y_2+2)+(y_3+2)+(y_4+2)+(y_5+2)=24$, or $y_1+y_2+y_3+y_4+y_5=14$. We can find out how many solutions this has in essentially the same way as shown in the previous part, but this time with 14 items and 4 slots, yielding $\binom{18}{4}=3060$ solutions.

(c) *What if each x_i must be even?*

It would be useful to convert this to a problem already solved, and we may indeed do so by letting $x_i=2y_i$; then x_i is a nonnegative even integer if and only if y_i is a nonnegative integer. Solutions to $x_1+x_2+x_3+x_4+x_5=24$ are simply solutions to $2y_1+2y_2+2y_3+2y_4+2y_5=24$, or $y_1+y_2+y_3+y_4+y_5=12$. We can find out how many solutions this has in essentially the same way as shown in part (a), but this time with 12 items and 4 slots, yielding $\binom{16}{4}=1820$ solutions.

2.4.8. (a) *A small library contains 15 different books. If five different students simultaneously check outone book each, how many different book selections are possible?*

Since 15 distinguished items are our pool, and five distinct emlements are being selected distinguishably, this is a permutation statistic, and we get $P(15,5)=360360$ possible selections.

(b) *A catalog contains 15 different books. If five different students each order one book from the catalog (so repetitions are possible), how many different book selections are possible?*

Since 15 distinguished items are our pool, and five not necessarily distinct elements are being selected distinguishably, such a selection is essentially a function mapping $\{1,2,3,4,5\}$ to $\{1,2,\ldots,15\}$, and we get $15^5=759375$ possible selections.

2.4.10. *If 12 different candy bars are distributed to four children, what is the probability that the smallest child receives either one or two candy bars?*

The sample space of all events has size 4^{12}: we can see this since, for each of the distinguishable candy bars, we have 4 choices of recipient (note that, in more combinatorial terms, we are selecting 12 elements with order and with repetition from a pool of 4 possibilities). The space of desirable events we will divide into two cases: either the smallest child receives one candy or two.

The number of ways in which this child receives a single candy can be enumerated as follows: a candy is chosen for the smallest child, and then the remainder are distributed to the three remaining children. The smallest child can receive any of 12 different candies, and then the distribution of 11 candies among 3 children can proceed any of 3^{11} different ways, so this case contributes $12\cdot 3^{11}$ possible distributions.

If the smallest child receives two candies, then we once again select the smallest child's candies first, then distribute the remainder. The former chicce is a selection of two distinct items from a pool of 12: thus, it is $\binom{12}{2}$. Our latter choice involves distributing 10 distinguishable items to three people, which, much as above can proceed in any of 3^{10} ways. Thus this case accounts for $\binom{12}{2}3^{10}$ possibilities.

Adding up the cases, we see that there are $12 \cdot 3^{11} + \binom{12}{2}3^{10}$ distributions in which the smallest child receives one or two candies. Thus, to compute a probability, we divide by the set of all distributions to get:

$$\frac{12 \cdot 3^{11} + \binom{12}{2}3^{10}}{4^{12}} = \frac{102 \cdot 3^10}{4^12} \approx 36\%$$

2.4.14. *How many different functions are there from a set A, with n different elements, to a set B, with $n+2$ elements? How many injections are there from A to B? How many surjections are there from B to A?*

If we freely choos functions, then each element of A can be mapped to any element in B, so for each of the n elements of A, there are $n+2$ choices of image; thus overall there are $(n+2)^n$ choices of function.

If we restrict to injections, then we may may the first element of A to any element in B, but the second element in A must be mapped to one of the remaining $n+1$ elements of B, and so forth. This is identical to choosing n distinct elements in order from B, so it is exactly the permutation statistic $P(n+2, n) = \frac{(n+2)!}{2}$

Surjections from B to A are more complicated, because there are two possibilities to contend with: there must be n distinct images in A so as to be a surjection, but, the two extra elements of B can be distributed either to a single element of A, or among two different elements. We must analyze these cases separately:

Suppose there is an element of A which is the image of 3 seperate members of B. To enumerate functions with this property, we start by choosing the element of A which has 3 preimages: there are n ways to select this. Then we select the three elements of B which map to it, which we can do $\binom{n+2}{3}$ ways. Finally, the remaining elements $n-1$ elements of A and of B must be associated with each other in a one-to-one correspondence: this we can do $P(n-1, n-1) = (n-1)!$ ways. This case thus contributes $n\binom{n+2}{3}(n-1)! = \frac{n(n+2)!}{6}$ surjections.

If, instead, we have two elements of A which are the image of 2 seperate members of B each, our selection process is more complicated: We choose two elements of A with this property, which we can do in $\binom{n}{2}$ ways; then we associate each in turn with 2 members of B. We take one of the elements of A (arbitrarily, we might say, the larger, according to some ordering on the elements of A) and choose 2 preimages for it: this can be done in any of $\binom{n+2}{2}$ ways. Now we repeat for the smaller, but only n preimages remain, so we have $\binom{n}{2}$ possible selections for this one. Lastly, we associate the remaining $n-2$ elements of A and B with each other, in any of $(n-2)!$ ways. This case thus contributes $\binom{n}{2}\binom{n+2}{2}\binom{n}{2}(n-2)! = \frac{n(n-1)(n+2)!}{8}$ cases.

We thus have a total of $\frac{n(n+2)!}{6} + \frac{n(n-1)(n+2)!}{8} = \frac{(3n+1)(n+2)!}{24}$ surjections.

2.4.20. *Given n identical objects and n objects which are different from them and from each other, find the number of ways to select n objects out of these $2n$ objects.*

Let us consider a number of different cases concurrently. If we were to pick some number k of the identical objects, we could do so only one way (since any chosen k

of the identical objects looks the same as any other selection of k identical objects).; however, of the remaining $n-k$ objects, we are selecting them from apool of n objects, so there are $\binom{n}{n-k}$ ways to do this. Thus, for the case where exactly k of our objects are identical, there are $\binom{n}{n-k}$ ways to select. So to get the sum of these cases, we range over all possible values of k to get $\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{0}$ which we know to be 2^n. We may in fact get the 2^n directly by observing that selections of the type mentioned in the problem can be bijectively mapped to subsets of the distinct objects by simply removing all the objects from a selection drawn from the identical group or, inverting the mapping, to take a subset of size k and "top it off" by adding $n-k$ of the identical elements.

2.4.29. *The integer* 3 *can be expressed as an ordered sum of positive integers in four ways, namely,* 3, 2+1, 1+2, *and* 1+1+1. *Show that any positive integer n can be expressed as an ordered sum in 2^{n-1} ways.*

We could look at the ways to express n as an ordered sum of k positive integers; using a reduction like that used above in 2.4.4(b), this is the same as the number of ways to express $n-k$ as an ordered sum of k nonnegative integers. We thus place $k-1$ "walls" among $n-1$ "slots" to produce such an ordered partition, and can do so in $\binom{n-1}{k-1}$ ways. Collecting together the distinct cases $k=1$,$k=2$, …, $k=n$, we have the sum $\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1}$ which we know has sum 2^{n-1}.

One may also approach 2^{n-1} directly, by considering a line of n objects among which we shall place walls. Unlike in previous approaches using this technique, we do not allow two walls to be adjacent (since 0 is not a positive integer), and we allow as many walls as desired among the objects. Thus, we may spread out our n objects and consider the $n-1$ gaps between them. In each of these gaps we may choose to place or not place a wall, and these decisions describe a partition. Since we have $n-1$ decisions involved in this process,e ach with two choices, there are 2^{n-1} ways to partition n. Here is a visual example of all eight distinct partitions using these "walls" to decompose the number 4:

$\Diamond\ \Diamond\ \Diamond\ \Diamond$	4
$\Diamond\ \Diamond\ \Diamond \vert \Diamond$	$3+1$
$\Diamond\ \Diamond \vert \Diamond\ \Diamond$	$2+2$
$\Diamond\ \Diamond \vert \Diamond \vert \Diamond$	$2+1+1$
$\Diamond \vert \Diamond\ \Diamond\ \Diamond$	$1+3$
$\Diamond \vert \Diamond\ \Diamond \vert \Diamond$	$1+2+1$
$\Diamond \vert \Diamond \vert \Diamond\ \Diamond$	$1+1+2$
$\Diamond \vert \Diamond \vert \Diamond \vert \Diamond$	$1+1+1+1$