3.1.2. *A leap year is a year that is divisible by 4, but not by 100 unless it is also divisible by 400. How many leap years are there from 1988 through 2400, inclusive?*

There are 104 years divisible by 4 in that range, of which 4 are excluded since they are divisible by 100, and 2 are re-included as they are divisible by 400, so the count is $104 - 5 + 2 = 103$.

3.1.3. *Find the number of three-digit integers that are not divisible by 4, 5, or 6.*

Let X be the set of all three-digit integers (so $|X| = 900$), and let A, B, and C be respectively the sets of three-digit multiples of 4, 5, and 6. Then $A \cap B$, $A \cap C$, $B \cap C$, and $A \cap B \cap C$ are respectively the multiples of 20, 12, 30, and 60. Thus $|A| = 225$, $|B| = 180$, $|C| = 150$, $|A \cap B| = 45$, $|A \cap C| = 75$, $|B \cap C| = 30$, and $|A \cap B \cap C| = 15$. The sought value is the number of elements of X not in any of A, B, and C: in other words, $|\overline{A \cup B \cup C}|$, whic , by inclusion-exclusion, has size $900 - (225 + 180 + 150) + (45 + 75 + 30) - 15 = 480$.

3.1.12. (a) *In how many different orders can the letters of the word TENNESSEE be arranged?*

There are 4 Es, 2 Ss, 2 Ns, and one T. There are thus $\binom{9}{4,2,2,1} = 3780$ anagrams.

(b) *In how many of the arrangements are no two E's adjacent?*

There are several quantities it is easy to enumerate: the arrangements of letters freely, the arrangement of letters with 2 Es and a monolithic EE, the arrangements of letters with 1 E and a monolithic EEE, the arrangements of two monolithic EEs, or the arrangement of letters with just the monolithic EEEE. Unfortunately, some of these overcount; for instance, arrangements of 2 Es and an EE counts the enumeration TNNSSEEEE three times, counting the final block as EE E E, E EE E, and E E EE. Considering all the possible actual arrangements of Es, and how they appear in our enumerations, we may build the following table of number of times each situation is counted by a particular enumeration to aid us in preventing overcounts or undercounts:

Groupings of Es	4 Es	2 Es, 1 EE	2 EEs	1 E, 1 EEE	1 EEEE
All Es separate	1	0	0	0	0
One "EE" cluster	1	1	0	0	0
Two "EE" clusters	1	2	1	0	0
One "EEE" cluster	1	2	0	1	0
One "EEEE" cluster	1	3	1	2	1

We want an enumeration which *only* counts the very first row; so we want to determine, in essence, a linear combination of the columns of this table yielding the vector $(1, 0, 0, 0, 0)$. Either by inspection or by linear algebra, we can see that taking the first row, subtracting the second, adding the third and fourth, and subtracting the fifth yields the desired result. The rows are, in order, enumerated by $\binom{9}{4,2,2,1}$, $\binom{8}{2,1,2,2,1}$, $\binom{7}{2,2,2,1}$, $\binom{7}{1,1,2,2,1}$, and $\binom{6}{1,2,2,1}$. Thus, we can count the cases where all Es are separate with the linear combination

$$\binom{9}{4,2,2,1} - \binom{8}{2,1,2,2,1} + \binom{7}{2,2,2,1} + \binom{7}{1,1,2,2,1} - \binom{6}{1,2,2,1} = 450$$

Note: this can also actually be done using a direct approach. There are $\binom{6}{4} = 15$ ways to place Es, and $\binom{5}{2,2,1} = 30$ ways to fill the gaps with the remaining letters.

3.1.16. *Given $2n$ symbols, two each of n different types, how many arrangements are there with no pair of consecutive symbols the same?*

Let X be the universe of arrangements with no constraints; thus $|X| = \binom{2n}{2,2,\ldots,2}$, and let A_i consist of those arrangements where the i's are adjacent, so that arrangements with no adjacent identical symbols are in $\overline{A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n}$. Note that by considering the i's as a monolithic element, one can determine that each $|A_i| = \binom{2n-1}{1,2,2,\ldots,2}$; likewise, considering ii and jj as individual and distinguishable monolithic elements, $|A_i \cap A_j| = \binom{2n-2}{1,1,2,2,\ldots,2}$, and so forth, until $|A_1 \cap A_2 \cap \cdots \cap A_n| = \binom{2n-n}{1,1,1,1,\ldots,1}$. Assembling these for inclusion-exclusion, we see that:

$$|\overline{A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n}|$$
$$= \binom{2n}{2,2,\ldots,2} - \binom{n}{1}\binom{2n-1}{1,2,\ldots,2} + \binom{n}{2}\binom{2n-2}{1,1,2,\ldots,2} - \binom{n}{3}\binom{2n}{1,1,1}$$

3.1.17. *Find the number of integers from one to one million, inclusive, that are not a square, a cube, nor a fifth power.*

Let $X = \{1, 2, 3, \ldots, 1000000\}$. Let A be the subset of X consisting of perfect squares, B the subset consisting of perfect cubes, C the subset consisting of perfect fifth powers. Note that $A \cap B$ consists of perfect sixth powers, $A \cap C$ consists of perfect tenth powers, $B \cap C$ of perfect 15th powers, and $A \cap B \cap C$ of perfect 30th powers. Thus, for instance, $A = \{1^2, 2^2, \ldots, 1000^2\}$, $B = \{1^3, 2^3, \ldots, 100^3\}$, $C = \{1^5, 2^5, \cdots, 15^5\}$ (since $15^5 < 1000000 < 16^5$). Repeating for each set, we see that $|A \cap B| = 10$, $|A \cap C| = 3$, $|B \cap C| = 2$, and $|A \cap B \cap C| = 1$. Thus, by inclusion-exclusion, $|\overline{A \cup B \cup C}| = 1000000 - (1000 + 100 + 15) + (10 + 3 + 2) - 1 = 998899$

3.1.18. *Twelve different dice are rolled. How many outcomes will have at least one of each number 1, 2, 3, 4, 5, 6 showing?*

The total number of outcomes is 6^{12}; denote the set of all possible outcomes by $|X|$. For each i, let A_i consist of those outcomes which do not have an i rolled. Then the sought quantity is $|\overline{A_1 \cup A_2 \cup \cdots \cup A_6}|$. Note that each $|A_i|$ is 5^{12}, since the outcomes within A_i are those in which numbers except i are rolled, so there are 5 choices instead of 6. Likewise, $|A_i \cap A_j| = 4^{12}$, since two possible rolls are excluded; $|A_i \cap A_j \cap A_k| = 3^{12}$, and so forth. Thus, using the principle of inclusion-exclusion:

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_6}| = 6^{12} - \binom{6}{1}5^{12} + \binom{6}{2}4^{12} - \binom{6}{3}3^{12} + \binom{6}{4}2^{12} - \binom{6}{5}1^{12} + \binom{6}{6}0^{12}$$

This total is 953029440, which is also the number of surjections from a set of 12 elements to a set of 6 elements.

3.2.4. *How many secret codes can be made by assigning to each letter of the English language a unique letter that is different than itself?*

These are precisely the derangements of the alphabet, of which there are $\frac{26!}{0!} - \frac{26!}{1!} + \frac{26!}{2!} - \frac{26!}{3!} + \frac{26!}{4!} - \cdots + \frac{26!}{26!}$.

3.2.8. *Find the probability of rolling ten different dice and obtaining a sum of 30.*

The total number of possibilities if 6^{10}; the number of possibilities matching the event is exactly the number of solutions to $x_1+x_2+x_3+\cdots+x_{10} = 30$ such that each x_i is an integer from 1 to 6. We naturalize the lower bound by considering the transformation of solutions via $y_i = x_i - 1$, so that y_i is bounded between 0 and 5 and subject to the equality $y_1 + y_2 + \cdots + y_{10} = 20$. We know that there are $\binom{29}{9}$ solutions to this not subjected to the upper bound on y_i: to impose the upper bound, we shall systematically exclude those which violate it.

Let X be the set of all nonnegative integer solutions to $y_1 + \cdots + y_{10} = 20$; let A_i be the set of solutions to the aforementioned equation in which $y_i \geq 6$. Then, we may find that $|A_i| = \binom{23}{9}$ by the tranformation $z_i = y_i - 6$ to a single variable. Likewise, transforming two variables, we find that $|A_i \cap A_j| = \binom{17}{9}$, and $|A_i \cap A_j \cap A_k| = \binom{11}{9}$; intersections of more than three of these sets are empty, since forcing four or more numbers to be 6 or larger yields no nonnegative solutions to $y_1 + \cdots + y_{10} = 20$.

Now, we may calculate the number of elements of X not violating any of these conditions, that is to say, $|\overline{A_1 \cup A_2 \cup \cdots \cup A_{10}}|$ using inclusion-exclusion to be

$$\binom{29}{9} - \binom{10}{1}\binom{23}{9} + \binom{10}{2}\binom{17}{9} - \binom{10}{3}\binom{11}{9}$$

3.1.9. *A bag of coins contains nine pennies, six nickels, four dimes, and three quarters. Assuming that coins of any one denomination are identical, in how many ways can ten coins be selected?*

If we enumerate the number of pennies, nickels, dimes, and quarters with x_1, x_2, x_3, and x_4 respectively, the question asks how to choose nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 10$ subject to the conditions $x_1 \leq 9$, $x_2 \leq 6$, $x_3 \leq 4$, and $x_4 \leq 3$. Let our universe X consist of all nonnegative solutions, and let A_1, A_2, A_3, and A_4 correspond to the situations violating respective conditions, so A_1 consists of those solutions with $x_1 \geq 10$, A_2 those with $x_2 \geq 7$, A_3 those with $x_3 \geq 5$, and A_4 those with $x_4 \geq 4$. Note that $|X| = \binom{13}{3}$. Then, forcing 10 coins to be pennies, $|A_1| = \binom{3}{3}$; forcing 7 coins to be nickels, 3 remain to be classified, so $|A_2| = \binom{6}{3}$; likewise $|A_3| = \binom{8}{3}$, and $|A_4| = \binom{9}{3}$. The overlaps of these sets are minimal, since 10 coins are insufficient to simultaneously satisfy any two of $x_1 \geq 10$, $x_2 \geq 7$, and $x_3 \geq 5$. The only nontrivial intersection, in fact, is $A_3 \cap A_4$, which requires pre-emptive classification of 9 coins, leaving one to be classified in any of $\binom{4}{3}$ ways. Thus, the total number of solutions not violating any of these rules is $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = \binom{13}{3} - (\binom{3}{3} + \binom{6}{3} + \binom{8}{3} + \binom{9}{3}) + (\binom{4}{3}) = 129$.

3.1.14. *Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 0$ in integers between -4 and 4, inclusive.*

First we want to shift the entire situation to be bounded below by 0 instead of -4: let $y_i = x_i + 4$, so that solutions to $x_1 + x_2 + x_3 + x_4 = 0$ with $-4 \leq x_i \leq 4$ are bijectively mapped to solutions of $y_1 + y_2 + y_3 + y_4 = 16$ with $0 \leq y_i \leq 8$. Now, we find solutions to this using inclusion-exclusion: let X consist of all nonnegative solutions to $y_1 + y_2 + y_3 + y_4 = 16$; let A_i correspond to solutions in which $y_i \geq 9$. Then the desired enumeration here is those solutions not satisfying any of the A_i; that is, $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}|$. Note that, if we force $y_i \geq 9$, then we have a value of 7 remaining to allocate, so $|A_i| = \binom{10}{3}$. Furthermore, since we cannot allocate 9 units to two different y_i with a total of 16 units, we can see that $|A_i \cap A_j| = 0$, so our inclusion-exclusion becomes quite simple: $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = \binom{19}{3} - 4\binom{10}{3} = 609$.

3.1.26. *If n letters are placed at random into n envelopes, what is the average number of letters that get into the correct envelope?*

Instead of counting envelope-assignments with correct choices (as we did for the derangement problem), we are here counting the total number of correct envelopes, so if A_1 consists of all assignments with the first letter correctly placed and so forth, the total number of correct placements over all assignments will be $|A_1| + |A_2| + \cdots + |A_n|$. Here we do not need to exclude intersections because we *want* to count them twice: e.g. an assignment with both the first and second letters put in the correct envelope should be counted twice, to count the fact that two envelopes are correct.

Since each $|A_i| = (n-1)!$, we find that $|A_1| + \cdots + |A_n| = n(n-1)! = n!$. This is the total number of correct placements summed over all possible letter-assignments. To find the average number of correct letter placements, we divide this grand total by the nubmer of possible letter-assignments, to get $\frac{n!}{n!} = 1$.