3.3.4. *A juggler colors 12 identical juggling balls red, white, and blue.*

(a) *In how many ways can this be done if each color is used at least once?*

Let us preemptively color one ball in each color, so that the 9 remaining balls may be colored any way we like. This is a question of partitioning 9 identical items into 3 distinct classes, alternatively, it could be considered as a non-negative integer solution to the problem $x_1 + x_2 + x_3 = 9$. In either case, the correct enumerator is $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$.

(b) *In how many ways can this be done if each color is used at least three times?*

We proceed above, but instead we preemptively color three balls in each color, leaving only 3 balls left over to be freely colored; here our enumerator will thus be $\binom{3+3-1}{3-1} = \binom{5}{2} = 10$. We could in fact enumerate these manually: there are three colorings in which there are 3 balls of two colors and 6 of the remainder, six colorings in which there are 3, 4, and 5 balls in three distinct colors, and one coloring in which there are 4 balls of each color.

3.3.8. (a) *In how many ways can 12 identical coins be distributed to four different persons if each person receives at least one coin?*

This is much as in the previous problem: preemptively hand out one coin to each person, and then partition the remaining 8 coins freely. This is counted by $\binom{8+4-1}{4-1} = \binom{11}{3} = 165$.

(b) *In how many ways can 12 different books be distributed to four identical boxes if each box receives exactly three books?*

There are several approaches to this problem: one is to partition the books among distinguishable boxes in $\binom{12}{3,3,3,3}$ ways, and then, realizing that each partition is represented by 4! different arrangements of boxes, divide by 4! to get $\frac{12!}{3!3!3!3!4!}$ ways of arranging the books.

Alternatively, one may number the books (for instance) 1–12, and consider the boxing of the books as a selection process. We select book 1 and two other books to accompany it any of $\binom{11}{2}$ ways; then we take out the lowest-numbered book remaining, and associate it with two of the remaining 8 books in $\binom{8}{2}$ ways; then we again take the lowest-numbered book and box it with two of the remaining 5 books in $\binom{5}{2}$ ways; and lastly, we take the remaining lowest-numbered book, and box it up with the remaining 2 books in $\binom{2}{2}$ ways.

These two approaches, of course, yield the same result: $\frac{12!}{3!3!3!3!4!} = \binom{11}{2}\binom{8}{2}\binom{5}{2}\binom{2}{2} = 15400$.

(c) *Compute the number of ways to partition a set of 12 different objects into four nonempty subsets.*

The partition is not explicitly ordered and partition of a set into unordered subsets is the precise definition of the Stirling number, so here the quantity desired is simply

$$S(12,4) = \frac{1}{4!}\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^{12} = \frac{\binom{4}{0}4^{12} - \binom{4}{1}3^{12} + \binom{4}{2}2^{12} - \binom{4}{3}1^{12} + \binom{4}{4}0^{12}}{4!} = 611501$$

3.3.10. *Explain the following formulas:*

(a) $S(n, n-1) = \binom{n}{2}$.

There are two possible explanations: either an algebraic demonstration of equality or construction of a bijection between quantities enumerated by $S(n, n-1)$ and $\binom{n}{2}$. The algebra is possible but difficult; the combinatorial construction is actually far simpler. Note that $S(n, n-1)$ enumerates the partitions of $\{1, 2, 3, \ldots, n\}$ into $n-1$ nonempty, unordered subsets. Such a partition would consist of $n-2$ sets of size 1 and one set of size 2. Thus, the partitions are uniquely determined by the choice of two-element set, since the rest of any given partition can be reconstructed by placing every element of $\{1, 2, \ldots, n\}$ not appearing in the two-element set in a singleton set by itself. Thus, nonempty partitions of an n-element set into $(n-1)$ subsets, which are enumerated with $S(n, n-1)$ are in a bijective correspondence to choices of a single two-element subset of an n-element set, which are enumerated with $\binom{n}{2}$. Thus, $S(n, n-1) = \binom{n}{2}$.

(b) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$.

The combinatorial proof of this follows closely the surjection argument from problem 2.4.14 (appearing on problem set #3). Here $S(n, n-2)$ enumerates partitions of $\{1, 2, 3, \ldots, n\}$ into $n-2$ nonempty, unordered subsets. This can occur in two different ways: either we partition $\{1, 2, 3, \ldots, n\}$ into $n-3$ singletons and a triplet, or into $n-4$ singletons and two pairs. Let A and B be sets consisting of these two types of partitions respectively : it is clear that $A \cap B = \emptyset$ and that $A \cup B$ is the set of all partitions of $\{1, 2, 3, \ldots, n\}$ into $n-2$ subsets. Thus, $S(n, n-2) = |A \cup B| = |A| + |B|$, and our aim is to bijectively map A and B onto sets enumerated with $\binom{n}{3}$ and $3\binom{n}{4}$.

A partition in A is uniquely determined by its triplet, since the remainder of the partition can be reconstructed by placing every element of $\{1, \ldots, n\}$ not appearing in the triplet into a singleton. Thus, elements of A can be bijectively mapped to 3-element subsets of $\{1, \ldots, n\}$, demonstrating that $|A| = \binom{n}{3}$.

A partition in B is uniquely determined by the two pairs occurring therein, since, as above, the partition can be reconstructed by placing every element not occurring in the pairs in a singleton. We may enumerate B thus by counting the number of pairs of pairs. If we choose four elements to belong to these pairs, we may do so in $\binom{n}{4}$ ways; but then, the assignment of 4 elements to two pairs may occur in any of 3 ways: $\{a, b\}$ and $\{c, d\}$; $\{a, c\}$ and $\{b, d\}$; or $\{a, d\}$ and $\{b, c\}$. Thus, there are $3\binom{n}{4}$ ways to choose two pairs as extraordinary parts of a partition.

3.3.16. (a) *In how many ways can n identical objects be distributed to five different boxes if the first two boxes receive no more than two balls each?*

The assignments of n identical objects to 5 boxes is equivalent to the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = n$$

which is $\binom{n+4}{4}$.

Now, we must exclude those solutions in which $x_1 \geq 3$ or $x_2 \geq 3$. We can enumerate the solutions where, for instance, $x_1 \geq 3$ by letting $x_1 = y_1 + 3$ and finding non-negative integer solutions to

$$\begin{aligned}(y_1+3)+x_2+x_3+x_4+x_5 &= n\\ y_1+x_2+x_3+x_4+x_5 &= n-3\end{aligned}$$

of which there are $\binom{n+1}{4}$. A similar argument finds the number of solutions in which $x_2 \geq 3$. Then, to find cases where both exceed 2, we use transformations on both x_1 and x_2 to find that these solutions are identically the non-negative solutions of

$$\begin{aligned}(y_1+3)+(y_2+3)+x_3+x_4+x_5 &= n\\ y_1+y_2+x_3+x_4+x_5 &= n-6\end{aligned}$$

of which there are $\binom{n-2}{4}$. Since we now know the total number arrangements, and the number of arrangements violating our conditions singly and in pairs, we may use inclusion-exclusion to find the number of arrangements violating none of our conditions:

$$\binom{n+4}{4} - \binom{n+1}{4} - \binom{n+1}{4} + \binom{n-2}{4} = \frac{9n^2-9n+12}{2}$$

(b) *In how many ways can n different objects be distributed to five different boxes if the first two boxes receive no more than two balls each?*

The most straightforward approach to this (that I can devise) is a casewise consideration: there are nine possibilities for the first two boxes containing anywhere from zero to four objects, and the remainder of the objects are distributed among the remaining 3 boxes. A table summarizing calculations associated with these nine cases (which can be reduced to 6 using symmetry) is below:

Box 1	Box 2	# selections	# remainder placements	Total configurations
0	0	$\binom{n}{0}\binom{n}{0}$	3^n	3^n
0	1	$\binom{n}{0}\binom{n}{1}$	3^{n-1}	$n3^{n-1}$
1	0	$\binom{n}{1}\binom{n-1}{0}$	3^{n-1}	$n3^{n-1}$
1	1	$\binom{n}{1}\binom{n-1}{1}$	3^{n-2}	$n(n-1)3^{n-2}$
0	2	$\binom{n}{0}\binom{n}{2}$	3^{n-2}	$\frac{n(n-1)}{2}3^{n-2}$
2	0	$\binom{n}{2}\binom{n-2}{0}$	3^{n-2}	$\frac{n(n-1)}{2}3^{n-2}$
1	2	$\binom{n}{1}\binom{n-1}{2}$	3^{n-3}	$\frac{n(n-1)(n-2)}{2}3^{n-3}$
2	1	$\binom{n}{2}\binom{n-2}{1}$	3^{n-3}	$\frac{n(n-1)(n-2)}{2}3^{n-3}$
2	2	$\binom{n}{2}\binom{n-2}{2}$	3^{n-4}	$\frac{n(n-1)(n-2)(n-3)}{2}3^{n-4}$

which has a total of

$$3^n + 2n3^{n-1} + 2n(n-1)3^{n-2} + n(n-1)(n-2)3^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2}3^{n-4}.$$

This can be cleaned up to become

$$\left(\frac{n^4}{2} + \frac{29n^2}{2} + 39n + 81\right)3^{n-4}.$$

3.R.1 *Among all n-digit integers, how many of them contain the digits 0 and 1 but not the digits 8 and 9?*

Let our universe set X consist of all n-digit integers consisting of digits $0 - -7$. Clearly $|X| = (7)(8)(8)\cdots(8) = 7 \cdot 8^{n-1}$. Now let A and B be the sets of n-digit integers consisting of digits $0 - -7$ respectively without 0 and without 1. Having both a 0 and a 1 is thus identical to membership in $\overline{A \cup B}$ (i.e. lacking neither a 0 nor a 1). Clearly $|A| = (7)(7)(7)\cdots(7) = 7^n$ and $|B| = (6)(7)(7)\cdots(7) = 6 \cdot 7^{n-1}$; and furthermore, $|A \cap B| = (6)(6)(6)\cdots(6) = 6^n$. Thus, our desired enumeration $|\overline{A \cup B}|$ can be calculated to be

$$|X| - |A| - |B| + |A \cap B| = 7 \cdot 8^{n-1} - 7^n - 6 \cdot 7^{n-1} + 6^n = 7 \cdot 8^{n-1} - 13 \cdot 7^{n-1} + 6^n$$

3.R.4 *How many ways are there to distribute 30 identical balls to six distinct boxes if box 1 and box 2 each receive fewer than 10 balls?*

This is much as in the first part of problem 3.3.16: the assignments of 30 identical balls to 6 distinct boxes is equivalent to the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$$

which is $\binom{35}{5}$.

Now, we must exclude those solutions in which $x_1 \geq 10$ or $x_2 \geq 10$. We can enumerate the solutions where, for instance, $x_1 \geq 10$ by letting $x_1 = y_1 + 10$ and finding non-negative integer solutions to

$$\begin{aligned}(y_1 + 10) + x_2 + x_3 + x_4 + x_5 + x_6 &= 30\\ y_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 20\end{aligned}$$

of which there are $\binom{25}{5}$. A similar argument finds the number of solutions in which $x_2 \geq 10$. Then, to find cases where both exceed 10, we use transformations on both x_1 and x_2 to find that these solutions are identically the non-negative solutions of

$$\begin{aligned}(y_1 + 10) + (y_2 + 10) + x_3 + x_4 + x_5 + x_6 &= 30\\ y_1 + y_2 + x_3 + x_4 + x_5 + x_6 &= 10\end{aligned}$$

of which there are $\binom{10}{5}$. Since we now know the total number arrangements, and the number of arrangements violating our conditions singly and in pairs, we may use inclusion-exclusion to find the number of arrangements violating none of our conditions:

$$\binom{35}{5} - 2\binom{25}{5} + \binom{15}{5} = 221375$$

3.R.5 *A bridge hand is a subset of 13 cards chosen from the standard deck. How many bridge hands contain 4 cards of the same rank?*

Let A_i be the set of bridge hands which contain 4 is (e.g. A_1 has hands with 4 aces, A_2 consists of those with 4 twos, and so forth up to A_13 consisting of hands with four kings). Clearly each $|A_i| = \binom{48}{9}$, since 4 cards are predetermined and the remaining 9 cards of the hand are chosen freely from the remaining 48 cards of the deck. Likewise, $|A_i \cap A_j| = \binom{44}{5}$ and $|A_i \cap A_j \cap A_k| = \binom{40}{1}$. Intersections of more than 3 sets are obviously empty, since no hand of 13 cards consists of 4 cards in each of 4 different ranks.

Since there are $\binom{13}{1}$ choices of i for listing all A_i, and $\binom{13}{2}$ choices of i and j to list all possible $A_i \cap A_j$, and $\binom{13}{3}$ choices of i, j, and k to list all possible $A_i \cap A_j \cap A_k$, we may see by inclusion-exclusion that:

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_{13}| &= \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \\ &= \binom{13}{1}\binom{48}{9} - \binom{13}{2}\binom{44}{5} + \binom{13}{3}\binom{40}{1} = 21717689136\end{aligned}$$

Incidentally, the likelihood of this occurrence is approximately 3%.

7.1.2 *Construct a generating function for a_n, the number of distributions of n identical juggling balls to*

(a) *Six different jugglers with at most four balls distributed to each juggler.*

We want our exponent on z to record the number of juggling balls distributed by every step of our process. Let the individual steps be the act of giving balls to a single juggler. Since our balls are identical, our choices in a single step are: give no balls to the juggler (which can be done only one way), represented by the polynomial $1z^0$ (or just 1); give one ball (which can be done in only one way), represented by the polynomial $1z^1$, and so forth up to giving 4 balls to a juggler. Thus, the act of giving a juggler up to 4 balls is represented by the polynomial $1+z+z^2+z^3+z^4$. Doing so for each of six jugglers, we get $(1+z+z^2+z^3+z^4)^6$ as our generating function.

(b) *Five different jugglers with between three and seven balls (inclusive) distributed to each juggler.*

As above, but the act of giving each juggler a collection of balls is rtepresented by the polynomial $z^3 + z^4 + z^5 + z^6 + z^7$; so doing so for each of five jugglers has generating function $(z^3 + z^4 + z^5 + z^6 + z^7)^5$.

7.1.5 *Find a generating function for the number of integers whose digits sum to n among*

(a) *Integers from 0 to 9999.*

An easy representation for these is as four-digit number-sequences with leading zeroes allowed; so we represent 37 as 0037, for instance (note that this has no effect on the digit sum). We break our selection of a number down into the four steps of picking individual digits. Since the information we most want to record with each step is the digit sum, we use the value of a digit as the exponent of z. Thus, for each digit, we have ten choices zero through nine, and for each choice, we want to record the number of ways to make that choice (of which there is only one) as the coefficient, and the value of that choice as the exponent. We thus associate with choice of a digit the polynomial $1z^0 + 1z^1 + 1z^2 + \cdots + 1z^9$; doing so four times, we get the generating function $(1 + z + z^2 + \cdots + z^9)^4$.

(b) *Four-digit integers.*

This is as above, but with the restriction that our first digit cannot be zero; thus the first-digit selection polynomial omits the $1z^0$ term, giving us the generating function $(z + z^2 + \cdots + z^9)(1 + z + z^2 + \cdots + z^9)^3$.

7.1.6 *Find a generating function for the number of ways to select n balls from an infinite supply of red, white, and blue balls subject to the constraint that the number of blue balls selected is at least 3, the number of red balls selected is at most 4, and an odd number of white balls are selected.*

Since we want to record the number of balls selected, we shall use that as the exponent of z. Our selection can be broken up into 3 independent procedures: selection of red balls, white balls, and blue balls. Our selection of red balls demands no more than 4, so there are 5 possibilities: we can select zero, one, two, three, or four. This step is thus representable by the polynomial $1 + z + z^2 + z^3 + z^4$. The selection of white balls demands an odd number, so we may select one ball, or three, or five, and so forth: this yields the infinite series $z^1 + z^3 + z^5 + z^7 + \cdots$, since there is no constraint placed on how many white balls we take. Likewise, we may take any number of blue balls three or higher: we may take three, or four, or five, and so forth, yielding the series $z^3 + z^4 + z^5 + \cdots$ to describe this step. Multiplying all our steps gives us the generating function for the process as a whole:

$$(1 + z + z^2 + z^3 + z^4)(z + z^3 + z^5 + z^7 + \cdots)(z^3 + z^4 + z^5 + \cdots)$$

7.1.14 *Find a generating function for the number of positive integer solutions of*

(a) $2x_1 + 3x_2 + 4x_3 + 5x_4 = n$.

Let the exponent of z represent the total sum on the left side of the equation; then choice of x_1 as 1, 2, 3, and so forth would increase the left side by 2, or 4, or 6, and so forth, so the contribution of x_1-selection to the generating function is $z^2 + z^4 + z^6 + z^8 + \cdots$. Likewise selecting x_2 to be a positive integer increases our total by thrice its value, so selection of x_2 may yield an increase of any multiple of 3 in the total left-side sum, so x_2-selection is associated with the series

$z^3 + z^6 + z^9 + z^{12} + \cdots$. Likewise, x_4 and x_5 are associated with the respective series $z^4 + z^8 + z^{12} + z^{16} + \cdots$ and $z^5 + z^{10} + z^{15} + z^{20} + \cdots$.

(b) $x_1 + x_2 + x_3 + x_4 = n$, *where each* x_i *satisfies* $2 \leq x_i \leq 5$.

Here we can freely choose x_1 to be 2, 3, 4, or 5, which results in addition of 2, 3, 4, or 5 to the total sum being kept track of; thus, our decision procedure for x_1 is represented by the polynomial $z^2 + z^3 + z^4 + z^5$; likewise for x_2, x_3, and x_4, so our generating function is $(z^2 + z^3 + z^4 + z^5)$.