7.2.1. (a) *Find the coefficient of* z^k *in* $(z^4+z^5+z^6+z^7+\cdots)^5$, $k \geq 20$.

We use the known series expansion $\frac{1}{(1-z)^\ell} = \sum_{n=0}^{\infty} \binom{n+\ell-1}{\ell-1} z^n$ below:

$$\begin{aligned}(z^4+z^5+z^6+z^7+\cdots)^5 &= (z^4)^5(1+z+z^2+z^3+\cdots)^5 \\ &= z^{20}\left(\frac{1}{1-z}\right)^5 \\ &= z^{20}\sum_{n=0}^{\infty}\binom{n+5-1}{5-1} z^n \\ &= \sum_{n=0}^{\infty}\binom{n+4}{4} z^{n+20}\end{aligned}$$

This is an acceptable but not entirely canonical representation of the generating function: to be standardized (and useful for answering the question asked) we'd like to change the exponent on z to be a single variable. So let $k = n+20$, replacing every n above with $k-20$:

$$\sum_{k-20=0}^{\infty}\binom{(k-20)+4}{4} z^k = \sum_{k=20}^{\infty}\binom{k-16}{4} z^k$$

so the coefficient of z^k is $\binom{k-16}{4}$ for $z \geq 20$ (and for $z < 20$, the coefficient is zero, as these terms do not appear in the series expansion).

(b) *Find the coefficient of* z^k *in* $(z+z^3+z^5)(1+z)^n$, $k \geq 5$.

We use the known binomial expansion $(1+z)^n = \sum_{i=0}^{\infty}\binom{n}{i} z^i$ below. Note that the upper bound can be ∞, since every binomial coefficient $\binom{n}{k}$ with $k > n$ is zero; this is not necessary, but actually makes our series manipulation easier, since we need not worry about the upper limit of the sum:

$$\begin{aligned}(z+z^3+z^5)(1+z)^n &= (z+z^3+z^5)\sum_{i=0}^{\infty}\binom{n}{i} z^i \\ &= \sum_{i=0}^{\infty}\binom{n}{i} z^{i+1} + \sum_{i=0}^{\infty}\binom{n}{i} z^{i+3} + \sum_{i=0}^{\infty}\binom{n}{i} z^{i+5}\end{aligned}$$

Now we need to perform index-shifts on each sum to get our z-terms to match; so we introduce new indices $k = i+1$, $k = i+3$, and $k = i+5$ to the three sums:

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{n}{i} z^{i+1} + \sum_{i=0}^{\infty}\binom{n}{i} z^{i+3} + \sum_{i=0}^{\infty}\binom{n}{i} z^{i+5} \\ &\quad = \sum_{k=1}^{\infty}\binom{n}{k-1} z^k + \sum_{k=3}^{\infty}\binom{n}{k-3} z^k + \sum_{k=5}^{\infty}\binom{n}{k-5} z^k \\ &\quad = z + nz^2 + \left[\binom{n}{2}+1\right] z^3 + \left[\binom{n}{3}+n\right] z^4 + \sum_{k=5}^{\infty}\left[\binom{n}{k-1}+\binom{n}{k-3}+\binom{n}{k-3}\right] z^k\end{aligned}$$

So despite all the special-casing necessary for $k < 5$ (which need not be calculated as explicitly as shown here), we see that the coefficient of z^k for $k \geq 5$ will always be $\binom{n}{k-1} + \binom{n}{k-3} + \binom{n}{k-3}$.

7.2.6. *What is the generating function for a_n, the number of integers from 0 to 99,999, whose digits sum to n? How many of these integers have digits that sum to 27?*

The generating function for numbers from zero to 99,999 categorized by digit sum is $(1 + z + z^2 + \cdots + z^9)^5$ (cf. problem 7.1.5(a) from Problem Set #6). We shall perform expansions of the generating function to determine the z^{27} term of this series:

$$\begin{aligned}(1 + z + \cdots + z^9)^5 &= \left(\frac{1 - z^{10}}{1 - z}\right)^5 \\ &= (1 - z^{10})^5 \left(\frac{1}{1 - z}\right)^5 \\ &= (1 - 5z^{10} + 10z^{20} - 10z^{30} + 5z^{40} - z^{50}) \sum_{n=0}^{\infty} \binom{n+4}{4} z^n\end{aligned}$$

To find the z^{27} term, we consider all possible products of z^{27} that can be produced via a product of the 50th-degree polynomial and series given. A z^{27} term arises via all of the following products: $1 \cdot \binom{27+4}{4} z^{27}$, $-5z^{10} \cdot \binom{17+4}{4} z^{17}$, and $10z^{20} \cdot \binom{7+4}{4} z^7$. Thus, the total z^{27} coefficient in this product is:

$$\binom{31}{4} - 5\binom{21}{4} + 10\binom{11}{4} = 4840$$

7.2.11. *Show that the following formulas hold:*

(a) $\frac{1}{1-z} = (1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots$.

We can prove by induction that $(1+z)(1+z^2) \cdots (1+z^{2^n}) = 1+z+z^2+\cdots+z^{2^{n+1}-1}$: it is clearly true for $n = 1$, and assuming it for a particular n, we can derive its truth for $n + 1$:

$$\begin{aligned}(1 + z)(1 + z^2) \cdots (1 + z^{2^n}) &= 1 + z + z^2 + \cdots + z^{2^{n+1}-1} \\ (1 + z)(1 + z^2) \cdots (1 + z^{2^n})(1 + z^{2^{n+1}}) &= \left(1 + z + z^2 + \cdots + z^{2^{n+1}-1}\right)(1 + z^{2^{n+1}}) \\ &= \left(1 + z + z^2 + \cdots + z^{2^{n+1}-1}\right) \\ &\quad + \left(z^{2^{n+1}} + z^{2^{n+1}+1} + z^{2^{n+1}+1} + \cdots + z^{2^{n+1}+(2^{n+1}-1)}\right) \\ &= 1 + z + z^2 + \cdots + z^{2(2^{n+1})-1} \\ &= 1 + z + z^2 + \cdots + z^{(2^{n+2})-1}\end{aligned}$$

We thus have that $\prod_{i=0}^{n}(1 + z^{2^i}) = \sum_{i=0}^{2^{n+1}-1} z^n$, so in the limiting case (doing the sorts of things I tell MATH 205 students that we are never, ever allowed to do),

$$\prod_{i=0}^{\infty}(1 + z^{2^i}) = \sum_{i=0}^{\infty} z^n = \frac{1}{1 - z}$$

(b) $\frac{1}{1-z} = (1+z+z^2)(1+z^3+z^6)(1+z^9+z^{18})\cdots$.
This proceeds similarly to the argument above, only with 3^n instead of 2^n everywhere.

7.2.15. *Find a generating function and formula for a_n, the number of ways to distribute n similar juggling balls to four different jugglers so that each juggler receives an odd number of juggling balls that is larger than or equal to three.*

Monitoring the number of balls distributed, distribution of balls to a single juggler under these constraints can be modeled with the series $z^3 + z^5 + z^7 + \cdots = z^3(1 + z^2 + z^4 + \cdots) = \frac{z^3}{1-z^2}$. Doing so for 4 jugglers yields the generating function $\frac{z^{12}}{(1-z^2)^4}$.

To find a formula for a_n, we need to put this generating function in the form $\sum a_n z^n$ in order to make determining individual coefficients easy. We use the known series expansion for $\frac{1}{(1-y)^k}$ with z^2 standing in for y:

$$\frac{z^{12}}{(1-z^2)^4} = z^{12} \sum_{m=0}^{\infty} \binom{m+4-1}{4-1} (z^2)^m = \sum_{m=0}^{\infty} \binom{m+3}{3} z^{2m+12}$$

Now we use index-shifting, letting $k = m+6$ and replacing all occurrences appropriately to clean up the exponent on z and getting the following simplified sum:

$$\sum_{k-6=0}^{\infty} \binom{k-6+3}{3} z^{2k} = \sum_{k=6}^{\infty} \binom{k-3}{3} z^{2k}$$

Thus, the coefficient of z^{2k} is $\binom{k-3}{3}$, and the coefficient of any odd pwoer of z is zero; thus $a_n = \binom{k-3}{3}$ if $n = 2k$, and $a_n = 0$ if n is odd.

7.2.22. *In how many ways can a coin be flipped 25 times in a row so that exactly five heads occur and no more than seven tails occur consecutively?*

Let n keep track of the total number of flips in the following process: flip from zero to seven tails; flip a head then flip from zero to seven tails five times. This process is guaranteed to yield exactly five heads and no more than 7 consecutive tails. This procedure has 11 discrete steps, but they fall into two categories: flips of a single coin yielding heads, with generating function z; and flips of a coin from zero to seven times yielding tails, with generating function $1 + z + z^2 + \cdots + z^7$. Thus, our generating function in total is

$$(1 + z + z^2 + \cdots + z^7)\left[z(1 + z + z^2 + \cdots + z^7\right]^5 = \frac{z^5(1-z^8)^6}{(1-z)^6}$$

which can be calculated to be

$$z^5(1 - 6z^8 + 15z^{16} - \cdots + z^{48}) \sum_{n=0}^{\infty} \binom{n+5}{5} z^n$$

We then determine the z^{25} term in this series by finding all possible products of terms from the factors yielding z^{25}: $z^5 \cdot 1 \cdot \binom{25}{5} z^{20}$, $z^5 \cdot -6z^8 \cdot \binom{17}{5} z^{12}$, and $z^5 \cdot 15z^{16} \cdot \binom{9}{5} z^4$; thus the total coefficient of z^25 in the above product is $\binom{25}{5} - 6\binom{17}{5} + 15\binom{9}{5} = 17892$.

7.3.3. *Find a generating function for a_n, the number of partitions of n into*

(a) *Odd integers.*

A partition of n into odd numbers can specifically be considered as a partition into x_1 ones, x_3 threes, x_5 fives, etc.; then, these variables conform to the equation $x_1+3x_3+5x_5+\cdots = n$. The generating function describing the number of solutions to this equation is the product of the generating functions for the individual choices of x_1, x_3, x_5, etc. Thus, we have the product

$$(1+z+z^2+z^3+\cdots)(1+z^3+z^6+z^9+\cdots)(1+z^5+z^{10}+z^{15})\cdots = \frac{1}{(1-z)(1-z^3)(1-z^5)\cdots}$$

(b) *Distinct odd integers.*

This is as above, but with each x_i constrained to be zero or 1, so we have finite generating functions for each individual choice, yielding:

$$(1+z)(1+z^3)(1+z^5)(1+z^7)\cdots$$

7.3.4. *Find a product whose expansion can be used to find the number of partitions of*

(a) *12 with even summands.*

For even summands we are determining the number of nonnegative solutions to the equation $2x_2 + 4x_4 + 6x_6 + 8x_8 + 10x_{10} + 12x_{12} = 12$. The generating function for each of these can, in the interest of yielding a finite product, be capped at the z^{12} term, and their product is thus

$$(1+z^2+z^4+z^6+z^8+z^{10}+z^{12})(1+z^4+z^8+z^{12})(1+z^6+z^{12})(1+z^8)(1+z^{10})(1+z^{12})$$

whose product is an enormous 66th-degree polynomial, in which the only term of interest to us is $11z^{12}$ (so there are 11 even-term partitions of 12; unsurprising since each is the double of one of the 11 free partitions of 6).

(b) *10 with summands greater than 2.*

We are determining the number of nonnegative solutions to $3x_3+4x_4+5x_5+\cdots+10x_{10} = 10$; we do this by multiplying the generating functions for each summand, capping each series with the z^{10} term to get polynomials instead of series, and thus get

$$(1+z^3+z^6+z^9)(1+z^4+z^8)(1+z^5+z^{10})(1+z^6)(1+z^7)(1+z^8)(1+z^9)(1+z^{10})$$

whose product is a 67th-degree polynomial, in which we are interested in the term $5z^{10}$, denoting 5 partitions of 10 into parts 3 or larger; we can, knowing there are so few, actually explicitly enumerate them: $3+3+4$, $3+7$, $4+6$, $5+5$, and 10.

(c) *9 with distinct summands.*

We are determining the number of solutions to $x_1 + 2x_2 + 3x_3 + \cdots + 9x_9 = 9$ with each $x_i \in \{0, 1\}$. Thus the generating functions for individual summands are binomials, and we multiply them as such:

$$(1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5)(1+z^6)(1+z^7)(1+z^8)(1+z^9)$$

to get a 45th-degree polynomial which contains the term $8z^9$, indicating 8 partitions of 9 into distinct summands. The complete list of these is $1+2+6$, $1+3+5$, $2+3+4$, $1+8$, $2+7$, $3+6$, $4+5$, and 9.

7.3.6. (a) *Show that the number of partitions of $n-1$ is exactly equal to the number of partitions of n whose smallest summand is 1.*

One could do this with generating functions: we know that

$$(1+z+z^2+\cdots)(1+z^2+z^4+\cdots)(1+z^3+z^6+\cdots)\cdots = \sum_{n=0}^{\infty} p(n)z^n$$

but if we force our partitions to contain at least one "1", the first factor in the above product becomes $(z+z^2+z^3+\cdots) = z(1+z+z^2)$, yielding the generating function

$$z\sum_{n=0}^{\infty} p(n)z^n = \sum_{n=0}^{\infty} p(n)z^{n+1} = \sum_{n=1}^{\infty} p(n-1)z^n$$

So we see that forcing at least one 1 in a partition of n elements is possible in $p(n-1)$ ways.

We could also accomplish this with an explicit bijection. Any partition of n with a 1 in its expansion can be mapped to a partition of $n-1$ by simply removing the 1; this is clearly a bijective map since it can be reversed by taking any partition of $n-1$ and adding a single 1 to it. Since this is a bijection, the two sets involved must be of equal size.

(b) *Describe the partitions of n that are counted by the expression $p(n) - p(n-1)$.*

We know $p(n)$ counts the partitions of n; and from the above conclusion we know that $p(n-1)$ counts the partitions of n which have a 1 in their expansion. Thus, their difference counts the partitions of n which do not contain a 1 in their expansions.

Let P_n be the set of partitions of n, so that $|P_n| = p(n)$. Then $p(n) - p(n-1)$ would be the size of P_n with some set bijectively mapped to P_{n-1} removed. There is a simple injection of P_{n-1} into P_n: simply add a single "1" to every partition in P_{n-1} to get a partition from P_{n-1}. The set mapped onto by this operation is precisely those partitions of P_n containing "1" in their expansions, since this procedure could be reversed by removing a "1". The set yielded by removing all these images of P_{n-1} from P_n is thus those partitions of n not using the number 1; and this is what is counted by $p(n) - p(n-1)$.

7.3.7. *Use the Ferrer's graph to show that the number of partitions of n into exactly k summands equals the number of partitions of n having its largest summand equal to k.*

All one needs to do is consider the transpose. A Ferrer's graph associated with a partition into k summands will have height exactly k, and thus, its transpose will have width exactly k, meaning that one of its summands is size k but none exceed k. Since the transpose is a bijective map, the partitions satisfying these two conditions are equinumerous.

7.3.14. *Find a generating function for the number of partitions of n into*

(a) *Summands no larger than 4.*

We want nonnegative solutions to $x_1+2x_2+3x_6+4x_4=n$, which can be expressed as a product of the four individual generating functions for decision of x_1, x_2, x_3, and x_4:

$$(1+z+z^2+z^3+\cdots)(1+z^2+z^4+z^6+\cdots)(1+z^3+z^6+z^9+\cdots)(1+z^4+z^8+z^{12}+\cdots)$$

(b) *Summands the largest of which is 4.*

This is as above, except we require at least one 4, so we constrain $x_4 \geq 1$, which affects the fourth multiplicand in our expression:

$$(1+z+z^2+z^3+\cdots)(1+z^2+z^4+z^6+\cdots)(1+z^3+z^6+z^9+\cdots)(z^4+z^8+z^{12}+\cdots)$$

(c) *At most four summands.*

By the transposition bijection on Ferrer's diagrams we know this is an identical enumeration to part (a), and thus has the same generating function.

(d) *Exactly four summands.*

By the transposition bijection on Ferrer's diagrams we know this is an identical enumeration to part (a), and thus has the same generating function.