7.2.12. (a) *Find the coefficient of z^{2k} in $(1 + z^2 + z^4 + z^6 + \cdots)^n$.*

This geometric series can be rewritten as $\frac{1}{(1-z^2)^n}$, which has series expansion $\sum_{k=0}^{\infty} \binom{n+k-1}{n-1}(z^2)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^{2k}$, so the coefficient of z^{2k} is $\binom{n+k-1}{n-1}$.

(b) *Find the coefficient of z^{2k+1} in $(1 + z^2 + z^4 + z^6 + \cdots)(z^3 + z^5 + z^7 + \cdots)^3$.*

We factor the expression and apply known series representations as such:

$$\begin{aligned}
(1 + z^2 + z^4 + z^6 + \cdots)(z^3 + z^5 + z^7 + \cdots)^3 &= (1 + z^2 + z^4 + z^6 + \cdots)(z^3)^3(1 + z^2 + z^4 + \cdots)^3 \\
&= z^9(1 + z^2 + z^4 + \cdots)^4 \\
&= \frac{z^9}{(1-z^2)^4} \\
&= z^9 \sum_{n=0}^{\infty} \binom{n+4-1}{4-1} (z^2)^n \\
&= \sum_{n=0}^{\infty} \binom{n+3}{3} z^{2n+9}
\end{aligned}$$

To canonicalize the exponent of z (since $2n+9$ is a rather unwieldy and unconventional form), we choose k such that $2k+1 = 2n+9$; in other words, $k = n+4$. Substituting that into the above series,

$$\sum_{k-4=0}^{\infty} \binom{k-4+3}{3} z^{2(k-4)+9} = \sum_{k=4}^{\infty} \binom{k-1}{3} z^{2k+1}$$

so the coefficient of z^{2k+1} is $\binom{k-1}{3}$.

7.2.18. *Construct a generating function for a_n, the number of ways to distribute n juggling balls to four different jugglers if unlimited supplies of orange and white balls are available and each juggler receives at least two balls of each color.*

The process for this is an eightfold repetition of the act of handing at least 2 balls to a single juggler: first we hand juggler #1 white balls, then orange, then white to #2, and so forth. The generating function for each of these actions is $(z^2 + z^3 + z^4 + \cdots)$, so the generating function for the entire process is $\sum_{n=0}^{\infty} a_n x^n = (z^2 + z^3 + z^4 + \cdots)^8$. This is a complete answer to the question as given, but if you wish to determine particular values of a_n, the expression can be simplified as such:

$$\begin{aligned}
(z^2 + z^3 + z^4 + \cdots)^8 &= (z^2)^8(1 + z + z^2 + \cdots)^8 \\
&= \frac{z^{16}}{(1-z)^8} \\
&= z^{16} \sum_{k=0}^{\infty} \binom{k+8-1}{8-1} z^k \\
&= \sum_{k=0}^{\infty} \binom{k+7}{7} z^{k+16}
\end{aligned}$$

We then index-shift to produce a more canonical power-series form: $n = k + 16$, so

$$\sum_{n-16=0}^{\infty} \binom{(n-16)+7}{7} z^{(n-16)+16} = \sum_{n=16}^{\infty} \binom{n-9}{7} z^n$$

so a_n would be $\binom{n-9}{7}$.

7.4.4. *Find the number of n-digit ternary sequences that contain an odd number of 0's and an even number of 1's.*

This is a process consisting of three sorts of actions (choosing zeroes, ones, and twos), in which the order of the actions is significant; since we want to multiply generating functions representing different processes and get not only the number of different ways we can perform the processes but the number of ways to *order* performance of the processes, we use an exponential generating function, with the exponent on z recording how many terms of our sequence are being selected.

We have complete freedom to choose as many twos as we want, so our generating function for choice of twos is

$$1\left(\frac{z^0}{0!}\right) + 1\left(\frac{z^1}{1!}\right) + 1\left(\frac{z^2}{2!}\right) + 1\left(\frac{z^3}{3!}\right) + \cdots = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = e^z$$

Choosing to include zeroes instead, we are constrained to choose an odd number, so we only have odd-exponent terms:

$$1\left(\frac{z^1}{1!}\right) + 1\left(\frac{z^3}{3!}\right) + 1\left(\frac{z^5}{5!}\right) + 1\left(\frac{z^7}{7!}\right) + \cdots = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

This can be concisely expressed (if you are intimately familiar with Taylor series) as $\sinh z$ or as $\frac{e^z - e^{-z}}{2}$; it can also be shown to be $\frac{e^z - e^{-z}}{2}$ through manipulations of the known Taylor series for e^z and e^{-z}; which allow for cancellation of even terms upon subtraction.

Likewise, our calculation for inclusion of ones allows only an even number to be included, so we only use even exponents, yielding the generating function

$$1\left(\frac{z^0}{0!}\right) + 1\left(\frac{z^2}{2!}\right) + 1\left(\frac{z^4}{4!}\right) + 1\left(\frac{z^6}{6!}\right) + \cdots = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots$$

which, much like the above generating function, has a concise hyperbolic-function representation as $\cosh z$ or as the more familiar $\frac{e^z + e^{-z}}{2}$.

Thus, the generating function for a process in which these selection processed for zeroes, ones, and twos are commingled is:

$$\begin{aligned}\left(\frac{e^z - e^{-z}}{2}\right)\left(\frac{e^z + e^{-z}}{2}\right)e^z &= \left(\frac{e^{2z} - e^{-2z}}{4}\right)e^z \\ &= \frac{e^{3z} - e^{-z}}{4} \\ &= \frac{1}{4}\left(\sum_{n=0} \infty \frac{(3z)^n}{n!} - \sum_{n=0} \infty \frac{(-z)^n}{n!}\right) \\ &= \frac{1}{4}\left(\sum_{n=0} \infty 3^n \frac{z^n}{n!} - \sum_{n=0} \infty (-1)^n \frac{z^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \frac{3^n - (-1)^n}{4}\left(\frac{z^n}{n!}\right)\end{aligned}$$

Since this is the exponential generating function for the number of ways to select an ordered selection of n ternary digits (called "trits"), the number of ways to do so for any particular value of n is the coefficient of $\frac{z^n}{n!}$, which can be seen above to be $\frac{3^n-(-1)^n}{4}$.

7.4.6. *Find an exponential generating function for the number of ways to distribute n different objects to six different jugglers if each juggler receives between three and five objects.*

One can think of an assignment of distinct objects to n people as an n-ary sequence of digits; for instance, in this case the senary digit-string "1043552" would be an assignment of 7 distinct objects, giving object #2 to person zero, object #1 to person one, object #7 to person two, and so forth up to giving objects #5 and #6 to person five. As such, we can think of this selection process as a commingled selection of how many zeroes there are in the string, how many ones, etc. Each must appear between three and five times, so the exponential generating functions representing each individual selection is

$$1\left(\frac{z^3}{3!}\right) + 1\left(\frac{z^4}{4!}\right) + 1\left(\frac{z^5}{5!}\right) = \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120}$$

And the generating function for the process as a whole consists of six such processes, thus $\left(\frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120}\right)^6$. The expansion of this form is difficult to calculate and rather unilluminating. For example, the initial term is $\frac{z^{18}}{46656}$, more canonically represented as $137225088000\frac{z^{18}}{18!}$, which demonstrates that there are approximately 137 trillion distributions of a mere 18 objects under this scheme.

7.4.10. *Find the number of strings of length n that can be constructed using the alphabet $\{a, b, c, d, e\}$ if:*

(a) *b occurs an odd number of times.*

Free selection of any number of instances of a particular letter yields generating function $1\left(\frac{z^0}{0!}\right) + 1\left(\frac{z^1}{1!}\right) + 1\left(\frac{z^2}{2!}\right) + 1\left(\frac{z^3}{3!}\right) + \cdots = e^z$; selection of an odd number of instances of a particular letter yields $1\left(\frac{z^1}{1!}\right)+1\left(\frac{z^3}{3!}\right)+1\left(\frac{z^5}{5!}\right)+1\left(\frac{z^7}{7!}\right)+\cdots=\frac{e^z-e^{-z}}{2}$, as seen in problem 7.4.4. Thus, given exponential selection functions for all five letters which we want to mingle, we see that the generating function for this process as a whole is

$$\begin{aligned}(e^z)^4\left(\frac{e^z-e^{-z}}{2}\right)&=\frac{e^{5z}-e^{3z}}{2}\\&=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(5z)^n}{n!}-\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\frac{5^n-3^n}{2}\frac{z^n}{n!}\end{aligned}$$

so the number of ways to achieve an arrangement of n letters is the coefficient of $\frac{z^n}{n!}$, seen above to be $\frac{5^n-3^n}{2}$.

(b) *both a and b occur an odd number of times.* Using the same processes in the previous section, we get

$$\begin{aligned}(e^z)^3\left(\frac{e^z-e^{-z}}{2}\right)^2&=\frac{e^{5z}-2e^{3z}+e^z}{4}\\&=\frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(5z)^n}{n!}-2\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}+\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\frac{5^n-2\cdot 3^n+1}{4}\frac{z^n}{n!}\end{aligned}$$

so as above, the coefficient of $\frac{z^n}{n!}$ gives us our desired formula, $\frac{5^n-2\cdot3^n+1}{4}$.

7.4.16. *Find the exponential generating function for the number of ways to distribute n different objects to five people if person 1 receives fewer objects than person 2 and the total number of objects received by persons 1 and 2 is no more than 5.*

While a clever approach may be possible, the one which occurred most readily to me is brute force: produce an aggregate generating function for object selection for the first two people, then multiply by the easier-to-construct generating function for the remaining three. For the first two people, it is fairly straightforward to construct the generating function by hand, since it will only have terms up to $\frac{z^5}{5!}$ by construction. We look at the number of ways to distribute each number of objects zero through five to two people with person #1 receiving fewer objects than person #2: there are 0 ways to distribute zero objects, 1 way to distribute one (give it to #2), 1 way to distribute two (give both to #2), 4 ways to distribute three (give a single object to #1 in one of 3 different ways, or give them all to #2), 5 ways to distribute four (give a single object to #1 in one of 4 different ways, or give them all to #2), and 16 ways to distribute five (give zero, one, or two objects to #1 in 1, 5, or 10 ways respectively). Thus, distribution to the first two people has generating function

$$0\frac{z^0}{0!}+1\frac{z^1}{1!}+1\frac{z^2}{2!}+4\frac{z^3}{3!}+5\frac{z^4}{4!}+16\frac{z^5}{5!}$$

while the remaining three people, with freechoice, have the far more conventional exponential generating functions for their processes $1\left(\frac{z^0}{0!}\right)+1\left(\frac{z^1}{1!}\right)+1\left(\frac{z^2}{2!}\right)+1\left(\frac{z^3}{3!}\right)+\cdots=e^z$. Thus, the generating function for assignment to all 5 people is

$$\left(z+\frac{z^2}{2}+4\frac{z^3}{6}+5\frac{z^4}{24}+16\frac{z^5}{120}\right)e^{3z}$$

8.1.2. (a) *A man climbs a set of 10 steps, taking either 1 or 2 steps in each stride. In how many ways can he climb all 10 steps?*

Let a_n represent the number of such ways to climb n steps; then the above-requested quantity is a_{10}. To climb n steps, we have two choices: we can climb a single step and proceed somehow through the remaining $n-1$ steps, or we can climb two steps and then proceed somehow through the remaining $n-2$ steps. If we climb one step, we can climb the remaining $n-1$ steps in a_{n-1} ways, by the very definition of a_n; likewise, if we start by taking two steps, we may do the remainder in a_{n-2} ways. Thus, there are $a_{n-1}+a_{n-2}$ ways to climb n steps, so a_n is given by the recurrence $a_n=a_{n-1}+a_{n-2}$, and the initial conditions $a_0=1$ (since there is one valid way to climb zero steps: the null move) and $a_1=1$ (since there is one valid way to climb a single step: with a single one-step stride). We may use the recurrence to calculate the higher-valued a_i: $a_2=1+1=2$, $a_3=1+2=3$, $a_4=2+3=5$, $a_5=3+5=8$, $a_6=5+8=13$, $a_7=8+13=21$, $a_8=13+21=34$, $a_9=21+34=55$, and $a_{10}=34+55=89$. Thus there are 89 different traversals of the stairs. The sequence given by this recurrence is an index-shifted version of the Fibonacci sequence (OEIS A000045).

(b) *A man climbs a set of 10 steps, taking either 1, 2, or 3 steps in each stride. In how many ways can he climb all 10 steps?*

Following the same logic as above, we can demonstrate that, if the nubmer of ways to traverse n steps is denoted a_n, then this situation is described by the recurrence $a_n=a_{n-1}+a_{n-2}+a_{n-3}$, with initial values $a_0=1$, $a_1=1$, and $a_2=2$. Since we want a_{10}, we must calculate the intermediary values $a_3=1+1+2=4$, $a_4=1+2+4=7$, $a_5=2+4+7=13$, $a_6=4+7+13=24$, $a_7=7+13+24=44$, $a_8=13+24+44=81$, $a_9=24+44+81=149$, and $a_{10}=44+81+149=274$. Thus there are 274 possible traversals of the stairs. The sequence given by this recurrence is an index-shifted version of the so-called "Tribonacci" sequence (OEIS A000073).

8.1.6. *Find a recurrence relation (and initial conditions) for the number of ways to distribute n distinct objects to four distinct recipients.*

Let us denote the number of distributions of 4 distinct objects to n distinct recipients with a_n. We know, of course, that any formula we find for a_n should be identical to the closed-form formula $a_n=4^n$, since that is, from previous knowledge, what we know to be the number of free distributions of n distinct objects to four distinct recipients.

However, if we approach this from a recurrence-relation standpoint, we might divide the process of distributing n objects into two subprocesses: the distribution of the first

$n-1$ objects, and the distribution of the final object. The first $n-1$ objects can be distributed, by our definition of a_n, in a_{n-1} ways, and the final object can be distributed in 4 different ways (since it can be given to any of 4 recipients). Thus, the total number of ways to distribute n objects is $a_{n-1} \cdot 4$, leading to the recurrence $a_n = 4a_{n-1}$; together with the initial condition $a_0 = 1$ (since the only possible distribution of no objects is the null distribution), this does in fact yield the pattern we would expect to see, in which $a_n = 4^n$.

8.1.8. *Find a recurrence relation (and initial conditions) for the number of ternary sequences which:*

(a) *contain no consecutive 0's.*

Let us denote by a_n the number of ternary sequences of length n with no consecutive zeroes. Now, to build a recurrence, we would ask: how can I construct a ternary sequence of length n from a shorter sequence satisfying the above rules? A sequence ending in 1 or 2 is easy to construct: take a sequence of length $n-1$ and augment it with a 1 or a 2 on the end. We can construct $2a_{n-1}$ sequences ending in 1 or 2 this way, since there are a_{n-1} sequences of length $n-1$, each of which we can add a 1 or 2 to the end of. However, to find sequences of length n ending in 0, we must be cautious: we cannot in general augment a sequence of length $n-1$ by adding a 0 to the end. However, we may augment a sequence of length $n-2$ by adding 10 or 20 to the end safely. This yields $2a_{n-2}$ sequences of length n ending with 10 or 20. At this point we can be confident we've found all ternary sequences of length n without two consecutive zeroes, since we've found those ending in 1, in 2, in 10, or 20. The only suffix absent is 00, which we know will not occur. Thus, the number of sequences of length n satisfying the above conditions is $2a_{n-1}+2a_{n-2}$, so our recurrence is $a_n = 2a_{n-1}+2a_{n-2}$, and the initial conditions can be determined via brute force to be $a_0 = 1$ (the only zero-length sequence is the null sequence), $a_1 = 3$ (any of the sequences "0", "1", or "2"). This sequence is given as OEIS A028859; the property requested in this problem is given as the first comment in its description.

(b) *contain no blocks of three consecutive 0's.*

We proceed as above, determining a complete space of non-overlapping suffixes which we can add with impunity to a sequence with the abovementioned property to get a new sequence with the same property (for instance, above we had the suffixes 1, 2, 10, and 20, which was a complete set of possible suffixes, since the excluded suffix 00 could not possibly occur). As above we have 1, 2, 10, and 20, but in addition we may have two 0s occurring last, as long as they are preceded by a nonzero; thus we also have the safe suffix 100. At this point all suffixes have been identified except those with 000, which are of course forbidden. Thus, every "good" sequence (i.e. satisfying the given condition) of length n can be identified as one of the following:

- a good sequence of length $n-1$ with a 1 or 2 on the end,
- a good sequence of length $n-2$ with a 10 or 20 on the end, or

- a good sequence of length $n-3$ with a 100 or 200 on the end,

Thus, as above, we can calculate the recurrence for the number of good sequences of length n as given by $a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = 3$, and $a_2 = 9$ (the number of ternary sequences of length 0, 1, and 2). This sequence is given as OEIS A119826; the property requested in this problem is in fact its primary description.

8.1.10. *Find a recurrence relation for the number of ways to pair off $2n$ people for n simultaneous tennis matches.*

Let a_n be the number of ways to pair off $2n$ people. In calculating a_n, let us consider an individual (called Alice for the sake of illustration here). Alice must be paired with somebody, and since there are $2n-1$ people who are not Alice, she can be so paired in any of $2n-1$ ways. Then, we must pair off the $2(n-1)$ people who are neither Alice nor her partner. By the very definition of a_n, we may denote the number of ways to do so as a_{n-1}, Thus, the number of ways to perform this two-step process (pairing Alice, then pairing everyone else) is $(2n-1)a_{n-1}$, so $a_n = (2n-1)a_{n-1}$ forms our recurrence. Our initial condition is (as it so often is) $a_0 = 1$, since there is a trivial null pairing of zero people. This sequence is given as OEIS A001147; the property requested in this problem is given as the fourth comment in its description.

8.1.16. *Find a recurrence relation for the number of ways to pair off with nonintersecting lines $2n$ different points on a circle.*

Let us label our points, in order around the circumference, as $x_0, x_1, \ldots, x_{2n-1}$. Suppose x_0 is connected to some x_i. Then, from a topological point of view, we may think of the line x_0x_i as dividing the circle up into *two* circles: one with $x_1, x_2, \ldots, x_{i-1}$ about its circumference, and one with $x_{i+1}, x_{i+2}, \ldots, x_{2n-1}$ about its circumference. Thus, a pairing on $2n$ points which makes use of the edge x_0x_i can be constructed via a two-step process: perform a pairing on the points $x_1, x_2, \ldots, x_{i-1}$, and perform a pairing on $x_{i+1}, x_{i+2}, \ldots, x_{2n-1}$. Clearly there are zero ways to do this when i is even, since we have an odd number of points in each pairing demanded; however, if i is an odd number $2k+1$, then we are pairing $2k$ points in the first circle and $2n-2k-2$ in the second, which we may represent as a_k and a_{n-k-1}. Thus, there are a_ka_{n-k-1} ways to pair off the points if one of our lines is x_0x_{2k+1}. We may consider each value of k from zero to $n-1$ as a seperate case, so there are $\sum_{k=0}^{n-1} a_ka_{n-k-1}$ ways to pair off $2n$ points in total, giving the recurrence:

$$a_n = \sum_{k=0}^{n-1} a_ka_{n-k-1}$$

with initial condition $a_0 = 1$. This sequence is the highly versatile and useful Catalan numbers (OEIS A000108; the property requested in this problem is given as the fifth comment in its description, however, there are many other useful interprestations of the Catalan numbers).