8.2.4. (a) *Find the specific solution of* $a_n = 2a_{n-1} + 15a_{n-2}$ *with initial conditions* $a_0 = 1$ *and* $a_1 = 2$.

The characteristic polynomial of this recurrence relation is $x^2 - 2x - 15$, which has roots -3 and 5, so the general solution to this recurrence is $a_n = k_1(-3)^n + k_2 5^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = \quad k_1 + \ k_2 \\ 2 = -3k_1 + 5k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{3}{8}$, $k_2 = \frac{5}{8}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \frac{3(-3)^n + 5(5^n)}{8}$$

(b) *Find the specific solution of* $a_n = 3a_{n-1} + 2a_{n-2}$ *with initial conditions* $a_0 = 1$ *and* $a_1 = 2$.

The characteristic polynomial of this recurrence relation is $x^2 - 3x - 2$, which has roots $\frac{3+\sqrt{17}}{2}$ and $\frac{3-\sqrt{17}}{2}$, so the general solution to this recurrence is $a_n = k_1\left(\frac{3+\sqrt{17}}{2}\right)^n + k_2\left(\frac{3-\sqrt{17}}{2}\right)^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = k_1 + k_2 \\ 2 = \dfrac{3+\sqrt{17}}{2}k_1 + \dfrac{3-\sqrt{17}}{2}k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{1}{2} + \frac{\sqrt{17}}{34}$, $k_2 = \frac{1}{2} - \frac{\sqrt{17}}{34}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^n$$

8.2.8. *Consider the recurrence relation* $a_n = -a_{n-2}$. *Show that* $a_n = i^n$ *and* $a_n = (-i)^n$ *are solutions. For initial conditions* $a_0 = b$ *and* $a_1 = c$, *find constants* k_1 *and* k_2 *such that* $a_n = k_1 i^n + k_2(-i)^n$ *satisfies these initial conditions.*

The characteristic polynomial is $x^2 + 1$, with roots i and $-i$, so it would thus follow that $a_n = i^n$ and $a_n = (-i)^n$ are solutions to the recurrence relation; alternatively, one could use direct verification to show that these are solutions, as such: if $a_n = i^n$, then the recurrence $a_n = -a_{n-2}$ is clearly satisfied since $a_n = i^n = i^2 i^{n-2} = -i^{n-2} = -a_{n-2}$, and likewise for $a_n = (-i)^n$.

Since the general solution is thus $a_n = k_1 i^n + k_2(-i)^n$, if we are given initial conditions $a_0 = b$ and $a_1 = c$, then the coefficients in the above are determined by the system of equations resulting from evaluation of the general solution at $n = 0$ and $n = 1$:

$$\begin{cases} b = \ \ k_1 + \ \ \ k_2 \\ c = ik_1 - ik_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{b-ci}{2}$ and $k_2 = \frac{b+ci}{2}$.

8.2.10. *A man visits a pastry shop each morning. During each visit he buys either one of two types of pastry costing one dollar each or one of three types of pastry costing two dollars each. Find and solve a recurrence for the number of ways to spend n dollars at the pastry shop (order matters).*

Let a_n represent the number of ways to, over the course of several days, spend n dollars. Let us break this process down into the subprocess of purchasing a pastry on the first day and spending the rest of the money on subsequent days. One could buy either of the two inexpensive pastries on the first day — there are two ways to do so, since there are two different pastries — and then go on to spend $(n-1)$ dollars in the subsequent days, in any of a_{n-1} ways (following from the definition of a_n as the number of ways to spend n dollars). There are thus $2a_{n-1}$ ways to spend n dollars if one starts by purchasing and inexpensive pastry. On the other hand, if one chooses to purchase an expensive pastry the first day, this first day's purchase could be done in three different ways, since there are three different inexpensive pastries, and then the remaining days would involve the expenditure of $(n-2)$ dollars, which can be done in a_{n-2} ways. Thus, one could spend n dollars in $3a_{n-2}$ ways if one starts with the purchase of an expensive pastry. Combining the two cases of starting with a one-dollar or two-dollar pastry, we see that there are $2a_{n-1} + 3a_{n-2}$ ways in total to spend n dollars; thus the recurrence describing the number of ways to spend n dollars is $a_n = 2a_{n-1} + 3a_{n-2}$. We also need initial conditions, however: how many ways are there to spend exactly zero dollars or exactly one dollar? Our only option with zero dollars is to buy nothing at all; there is thus only one way to spend \$0 and therefore $a_0 = 1$. If we have one dollar, we may only purchase a cheap pastry, and have two ways of doing so, and therefore $a_1 = 2$.

The recurrence we wish to solve is $a_n = 2a_{n-1}+3a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$. The characteristic polynomial of htis recurrence relation is $x^2 - 2x - 3$, which has roots -1 and 3, so the general solution to the recurrence is $a_n = k_1(-1)^n + k_2 3^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = \ \ \ \ \ k_1 + \ \ k_2 \\ 2 = -1k_1 + 3k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{1}{4}$, $k_2 = \frac{3}{4}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \frac{(-1)^n + 3(3^n)}{4}$$

8.2.13. *Find and solve a recurrence relation for the number of n-digit ternary sequences that have no adjacent 0's.*

Compare this to problem 8.1.8(a) appearing on problem set #8, or to OEIS A028859, the latter of which references even includes the formula $a_n = \frac{(1+\sqrt{3})^{n+2}-(1-\sqrt{3})^{n+2}}{4\sqrt{3}}$. Using the results from 8.1.8(a), we know the recurrence here is $a_n = 2a_{n-1}+2a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 3$. This recurrence has characteristic polynomial x^2-2x-2 with roots $1+\sqrt{3}$ and $1-\sqrt{3}$, so the general solution is $a_n = k_1(1+\sqrt{3})^n+k_2(1-\sqrt{3})^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = & k_1 + & k_2 \\ 3 = & (1+\sqrt{3})k_1 + & (1-\sqrt{3})k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{1}{2} + \frac{\sqrt{3}}{3}$, $k_2 = \frac{1}{2} - \frac{\sqrt{3}}{3}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right)(1+\sqrt{3})^n + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right)(1-\sqrt{3})^n$$

which, though differently presented than the formula provided by Deutsch, is algebraically equivalent.

8.2.14. *Solve the recurrence relation $a_0 = x = a_1$ and $a_n = a_{n-1}a_{n-2}$ for $n \geq 2$.*

This is not a linear recurrence relation, so it seems difficult to apply linear methods to. However, we may convert that multiplication to an addition by means of applying a logarithm throughout (we shall use the base x logarithm here, in order to preserve good initial conditions, although this process will work reasonably well with any base on the logarithm. Then:

$$\begin{aligned} a_n &= a_{n-1}a_{n-2} \\ \log_x(a_n) &= \log_x(a_{n-1}a_{n-2}) \\ \log_x(a_n) &= \log_x(a_{n-1}) + \log_x(a_{n-2}) \end{aligned}$$

so if we let $b_n = \log_x a_n$, the above is simply the recurrence $b_n = b_{n-1}+b_{n-2}$, with initial conditions $b_0 = \log_x a_0 = \log_x x = 1$ and $b_1 = \log_x a_1 = \log_x x = 1$, which is to say, the Fibonacci sequence shifted forwards by one term. We may solve this recurrence, or simply appeal to our pre-existing knowledge of the Fibonacci sequece, to get the formula for b_n:

$$b_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and since $b_n = \log_x a_n$, it follows that $a_n = x^{b_n}$ and thus

$$a_n = x^{\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

8.3.2. *Find the general solution of*

(a) $a_n = a_{n-1} + 6a_{n-2} + 2n$.

The associated linear homogeneous recurrence relation is $b_n = b_{n-1} + 6b_{n-2}$, which has characteristic polynomial $x^2 - x - 6$ with roots 3 and -2, so the general solution of the associated homogeneous polynomial is $b_n = k_1 3^n + k_2(-2)^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is a linear polynomial in n, we choose a template for the particular solution which is also a linear polynomial: $a_n^* = \ell_0 + \ell_1 n$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned}\ell_0 + \ell_1 n &= (\ell_0 + \ell_1(n-1)) + 6(\ell_0 + \ell_1(n-2)) + 2n\\ \ell_0 + \ell_1 n &= (7\ell_0 - 13\ell_1) + (7\ell_1 + 2)n\end{aligned}$$

Matching up coefficients, we thus have the system of equations:

$$\begin{cases}\ell_0 = 7\ell_0 - 13\ell_1\\ \ell_1 = 7\ell_1 + 2\end{cases}$$

which we can solve to find that $\ell_1 = -\frac{1}{3}$ and $\ell_0 = -\frac{13}{18}$. Thus, $a_n^* = -\frac{1}{3}n - \frac{13}{18}$ is a solution to this recurrence; to get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = -\frac{1}{3}n - \frac{13}{18} + k_1 3^n + k_2(-2)^n$$

(b) $a_n - 5a_{n-1} + 6a_{n-2} = 5^n$.

The associated linear homogeneous recurrence relation is $b_n - 5b_{n-1} + 6b_{n-2} = 0$, which has characteristic polynomial $x^2 - 5x + 6$ with roots 3 and 2, so the general solution of the associated homogeneous polynomial is $b_n = k_1 3^n + k_2 2^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is an exponential expression, we choose a template for the particular solution which is also an exponential expression: $a_n^* = \ell 5^n$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned}\ell 5^n - 5\ell 5^{n-1} + 6\ell 5^{n-2} &= 5^n\\ 25\ell 5^{n-2} - 25\ell 5^{n-2} + 6\ell 5^{n-2} &= 25 \cdot 5^{n-2}\\ 6\ell &= 25\\ \ell &= \frac{25}{6}\end{aligned}$$

Thus, $a_n^* = \frac{25}{6}5^n$ is a solution to this recurrence; to get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = \frac{25}{6}5^n + k_1 3^n + k_2 2^n$$

8.3.6. *Solve the recurrence relation* $a_n + 2a_{n-1} + 2a_{n-2} = 4^n$ *with* $a_0 = 2$, $a_1 = 4$.

The associated linear homogeneous recurrence relation is $b_n + 2b_{n-1} + 2b_{n-2} = 0$, which has characteristic polynomial $x^2 + 2x + 2$ with roots $-1+i$ and $-1-i$, so the general solution of the associated homogeneous polynomial is $b_n = k_1(-1+i)^n + k_2(-1-i)^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is an exponential expression, we choose a template for the particular solution which is also an exponential expression: $a_n^* = \ell 4^n$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned} \ell 4^n + 2\ell 4^{n-1} + 2\ell 4^{n-2} &= 4^n \\ 16\ell 4^{n-2} + 8\ell 4^{n-2} + 2\ell 4^{n-2} &= 16 \cdot 4^{n-2} \\ 26\ell 4^{n-2} &= 16 \cdot 4^{n-2} \\ \ell &= \frac{16}{26} = \frac{8}{13} \end{aligned}$$

Thus, $a_n^* = \frac{8}{13}4^n$ is a solution to this recurrence; to get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = \left(\frac{8}{13}\right)4^n + k_1(-1+i)^n + k_2(-1-i)^n$$

To get a solution satisfying the given initial conditions from this, we plug in the values $n = 0$ and $n = 1$ to get:

$$\begin{cases} 2 = \dfrac{8}{13} + k_1 + k_2 \\ 4 = \dfrac{32}{13} + (-1+i)k_1 + (-1-i)k_2 \end{cases}$$

which we can solve to find that $k_1 = \frac{9}{13} - \frac{19}{13}i$ and $k_2 = \frac{9}{13} + \frac{19}{13}i$. Our solution to this recurrence with initial conditions is thus

$$a_n = \left(\frac{8}{13}\right)4^n + \left(\frac{9}{13} - \frac{19}{13}i\right)(-1+i)^n + \left(\frac{9}{13} + \frac{19}{13}i\right)(-1-i)^n$$

8.3.8. *Solve the recurrence relation* $a_n + a_{n-1} + a_{n-2} + a_{n-3} = 2$ *with* $a_0 = 1$, $a_1 = 2$, $a_2 = 3$.

The associated linear homogeneous recurrence relation is $b_n + b_{n-1} + b_{n-2} + b_{n-3} = 0$, which has characteristic polynomial $x^3 + x^2 + x + 1 = (x^2+1)(x+1)$ with roots

-1, i, and $-i$, so the general solution of the associated homogeneous polynomial is $b_n = k_1(-1)^n + k_2 i^n + k_3(-i)^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is a constant, we choose a template for the particular solution which is also a constant: $a_n^* = \ell_0$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned} \ell_0 + \ell_0 + \ell_0 + \ell_0 &= 2 \\ 4\ell_0 &= 2 \\ \ell_0 &= \frac{1}{2} \end{aligned}$$

so a particular solution to this differential equation is the (rather silly) sequence $a_n^* = \frac{1}{2}$. To get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = \frac{1}{2} + k_1(-1)^n + k_2 i^n + k_3(-i)^n$$

To get a solution satisfying the given initial conditions from this, we plug in the values $n = 0$, $n = 1$, and $n = 2$ to get:

$$\begin{cases} 1 = \frac{1}{2} + k_1 + \ k_2 + \ k_3 \\ 2 = \frac{1}{2} - k_1 + ik_2 - ik_3 \\ 3 = \frac{1}{2} + k_1 - \ k_2 - \ k_3 \end{cases}$$

which we can solve to find that $k_1 = \frac{3}{2}$, $k_2 = \frac{-1-3i}{2}$, and $k_3 = \frac{-1+3i}{2}$ $k_2 = \frac{9}{13} + \frac{19}{13}i$. Our solution to this recurrence with initial conditions is thus

$$a_n = \frac{1}{2} + \frac{3}{2}(-1)^n + \left(\frac{-1-3i}{2}\right) i^n + \left(\frac{-1+3i}{2}\right)(-i)^n$$

8.3.10. *Find a formula for* $\sum_{i=0}^{n} i^5$.

This can be rephrased as a recurrence relation quite easily: if a_n is the sum of the first n fifth powers, then we attain a_n from a_{n-1} merely by adding n^5; that is to say, $a_n = a_{n-1} + n^5$. To complete our recurrence we note the initial condition $a_0 = 0$. Now, to get a closed-form formula for this, we need only to solve this linear non-homogeneous recurrence.

The associated linear homogeneous recurrence relation is $b_n = b_{n-1}$, which has characteristic polynomial $x - 1$ with root 1, so the general solution of the associated homogeneous polynomial is $b_n = k_1$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is a fifth-degree polynomial, we choose a template for the particular solution which is also a fifth-degree polynomial: $a_n^* = \ell_0 + \ell_1 n + \ell_2 n^2 + \cdots + \ell_5 n^5$. Since this overlaps our homogeneous solution in a term (ℓ_0 and k_1 are both constant terms), we reject it and multiply our putative particular solution by n to try again with $a_n^* = \ell_0 n + \ell_1 n^2 + \ell_2 n^3 + \cdots + \ell_5 n^6$, which shares no terms with the homogeneous solution, so we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned}
\ell_0 n + \ell_1 n^2 + \ell_2 n^3 + \ell_3 n^4 + \ell_4 n^5 + \ell_5 n^6 &= \ell_0(n-1) + \ell_1(n-1)^2 + \ell_2(n-1)^3 + \ell_3(n-1)^4 + \ell_4(n-1 \\
\ell_0 n + \ell_1 n^2 + \ell_2 n^3 + \ell_3 n^4 + \ell_4 n^5 + \ell_5 n^6 &= (-\ell_0 + \ell_1 - \ell_2 + \ell_3 - \ell_4 + \ell_5) \\
&\quad + (\ell_0 - 2\ell_1 + 3\ell_2 - 4\ell_3 + 5\ell_4 - 6\ell_5)n \\
&\quad + (\ell_1 - 3\ell_2 + 6\ell_3 - 10\ell_4 + 15\ell_5)n^2 \\
&\quad + (\ell_2 - 4\ell_3 + 10\ell_4 - 20\ell_5)n^3 \\
&\quad + (\ell_3 - 5\ell_4 + 10\ell_5)n^4 \\
&\quad + (\ell_4 - 6\ell_5 + 1)n^5 \\
&\quad + \ell_5 n^6
\end{aligned}$$

so, with the exception of the trivial comparison of $\ell_5 n^6$ to $\ell_5 n^6$, a matching of coefficients here yields a system of five equations in five unknowns, which is not as bad as it looks, since the equations can be solved from the bottom up:

$$\begin{cases}
0 = -\ell_0 + \ell_1 - \ell_2 + \ell_3 - \ell_4 + \ell_5 \\
\ell_0 = \ell_0 - 2\ell_1 + 3\ell_2 - 4\ell_3 + 5\ell_4 - 6\ell_5 \\
\ell_1 = \ell_1 - 3\ell_2 + 6\ell_3 - 10\ell_4 + 15\ell_5 \\
\ell_2 = \ell_2 - 4\ell_3 + 10\ell_4 - 20\ell_5 \\
\ell_3 = \ell_3 - 5\ell_4 + 15\ell_5\ell_4 \qquad = \ell_4 - 6\ell_5 + 1
\end{cases}$$

so we can determine that $\ell_5 = \frac{1}{6}$, $\ell_4 = 3\ell_5 = \frac{1}{2}$, $\ell_3 = \frac{5}{2}\ell_4 - 5\ell_5 = \frac{5}{12}$, $\ell_2 = 2\ell_3 - \frac{10}{3}\ell_4 + 5\ell_5 = 0$, $\ell_1 = \frac{3}{2}\ell_2 - 2\ell_3 + \frac{5}{2}\ell_4 - 3\ell_5 = -\frac{1}{12}$, and $\ell_0 = \ell_1 - \ell_2 + \ell_3 - \ell_4 + \ell_5 = 0$. Thus, a specific solution to the above nonhomogeneous recurrence relation is

$$a_n^* = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + \frac{1}{12}n^2$$

and the general solution is

$$a_n = a_n^* + b_n = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + \frac{1}{12}n^2 + k_0$$

and we solve for k_0 by substituting in the case $n = 0$, yielding $0 = k_0$, so in this case $a_n = a_n^*$.

8.3.18. *What is the smallest amount that one can invest at interest rate i compounded annually in order that the person may withdraw one dollar at the end of the first year, four dollars at the end of the second year, and n^2 dollars at the end of the nth year in perpetuity?*

Note that in this question, i refers not to the imaginary quantity, but to some rate of interest.

Your bank balance at the end of the nth year under such a scheme would be your balance in the $(n-1)$th year multiplied by $(1+i)$ to represent interest earnings, with n^2 subtracted to represent the withdrawl. Thus, if your bank balance at the ned of the nth year is represented by a_n, the recurrence governing this process is $a_n = (1+i)a_{n-1} - n^2$. The question above is thus identical to asking wnhat choice of a_0 (i.e. your initial deposit) guarantees that all the terms of this recurrence are positive (the bank account is not exhausted). We need to solve the recurrence relation to answer this question.

The associated linear homogeneous recurrence relation is $b_n = (1+i)b_{n-1}$, which has characteristic polynomial $x - (1+i)$ with root $1+i$, so the general solution of the associated homogeneous polynomial is $b_n = k_1(1+i)^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is a quadratic, we choose a template for the particular solution which is also a quadratic: $a_n^* = \ell_0 + \ell_1 n + \ell_2 n^2$. Since this does not overlap our homogeneous solution in any of its terms (assuming $i > 0$), we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned} \ell_0 + \ell_1 n + \ell_2 n^2 &= (1+i)(\ell_0 + \ell_1(n-1) + \ell_2(n-1)^2) - n^2 \\ \ell_0 + \ell_1 n + \ell_2 n^2 &= (1+i)(\ell_0 - \ell_1 + \ell_2) + (1+i)(\ell_1 - 2\ell_2)n + ((1+i)\ell_2 - 1)n^2 \end{aligned}$$

Matching up coefficients, we thus have the system of equations:

$$\begin{cases} \ell_0 = (1+i)\ell_0 - (1+i)\ell_1 + (1+i)\ell_2 \\ \ell_1 = (1+i)\ell_1 - 2(1+i)\ell_2 \\ \ell_2 = (1+i)\ell_2 - 1 \end{cases}$$

which yields $\ell_2 = \frac{1}{i}$, $\ell_1 = \frac{2+2i}{i}\ell_2 = \frac{2+2i}{i^2}$, and $\ell_0 = \frac{1+i}{i}(\ell_1 - \ell_2) = \frac{2+3i+i^2}{i^3}$, so the recurrence is

$$a_n = \frac{n^2}{i} + \frac{2+2i}{i^2}n + \frac{2+3i+i^2}{i^3} + k_0(1+i)^n$$

solving for k_0 in terms of a_0, we see that $k_0 = a_0 - \frac{2+3i+i^2}{i^3}$, so the question becomes, what value of a_0 guarantees that

$$a_n = \frac{n^2}{i} + \frac{2+2i}{i^2}n + \frac{2+3i+i^2}{i^3} + \left(a_0 - \frac{2+3i+i^2}{i^3}\right)(1+i)^n$$

is positive for all n?