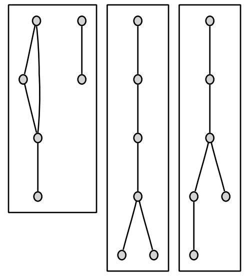
5.1.8. *Find three nonisomorphic graphs with the same degree sequence* $(1, 1, 1, 2, 2, 3)$.

There are several such graphs: three are shown below.



5.1.10. *Show that two projections of the Petersen graph are isomorphic.*

The isomorphism of these two different presentations can be seen fairly easily: pick any vertex from the first graph to associate with the center vertex of the second graph. Then, if we associate its neighbors in any order with the vertices at 12 o'clock, 4 o'clock, and 8 o'clock on the second representation, the association of the remaning 6 vertices should be reasonably straightforward.

Note: an explicit association is necessary. The fact that both graphs are conencted with every vertex of degree 3 is *not* sufficient — there are in fact 19 different connected 10-vertex graphs in which every vertex has degree 3. These graphs can all be seen at `http://www.mathe2.uni-bayreuth.de/markus/REGGRAPHS/10_3_3.html`.

5.1.12. *Suppose* G *is a graph with* p *vertices and* q *edges whose smallest degree is* δ^- *and whose largest degree is* δ^+. *Show that* $\delta^- \leq \frac{2q}{p} \leq \delta^+$.

For any vertex v in the graph, we know that $\delta^- \leq \deg v \leq \delta^+$. If we sum this inequality over all vertices v, we get

$$\sum_{v \in V(G)} \delta^- \leq \sum_{v \in V(G)} \deg v \leq \sum_{v \in V(G)} \delta^+$$

Since the argument of the sums in the lower and upper bounds are constant, these summations simply serve to multiply their arguments by $|V(G)| = |G| = p$; likewise, the middle term in the enequality we know to be $2\|G\| = 2q$. The above inequality can thus be simplified to

$$p\delta^- \leq 2q \leq p\delta^+$$

And thus $\delta^- \leq \frac{2q}{p} \leq \delta^+$.

5.1.13. *At the beginning of a math department meeting, there are 10 persons and a total of 26 handshakes.*

We shall represent this situation by way of a graph in which people are represented by vertices and handshakes between people are represented by edges between those vertices. Thus, this situation is described by a graph G with 10 vertices and 26 edges.

(a) *Explain why there must be a person who shakes hands at least six times.*

In the graph being used to represent this problem, there are 10 vertices and 26 edges. By the result determined in exercise 5.1.12, we know that in G, $\delta^+ \geq \frac{2\cdot\|G\|}{|G|} = \frac{2\cdot 26}{10} = 5.2$. Since δ^+ must be an integer, we thus know that δ^+ is at least 6, so there is some vertex in G with degree 6. Translating the edges incident to a vertex into handshakes in which the associated person has taken part, we see that the person associated with this vertex has shaken at least 6 hands.

(b) *Explain why there are not exactly three people who shake an odd number of hands.*

We know that $\sum_{v\in V(G)} \deg v = 2\cdot\|G\| = 52$. Let us partition $V(G)$ into two sets: put vertices of odd degree into A, and even degree into B. Then

$$52 = \sum_{v\in V(G)} \deg v = \sum_{v\in A} \deg v + \sum_{v\in B} \deg v$$

Suppose there are exactly three people who shake an odd number of hands, so $|A| = 3$. Then $\sum_{v\in A} \deg v$ is a sum of three odd nubmers, so $\sum_{v\in A} \deg v$ is odd. In order for the sum of $\sum_{v\in A} \deg v$ and $\sum_{v\in B} \deg v$ to be 52 (an even number), $\sum_{v\in B} \deg v$ must also be odd. However, by the definition of B, $\sum_{v\in B} \deg v$ is a sum of even numbers, which must be even, leading to a contradtction.

5.1.20. *During a summer vacation, nine students promise to keep in touch by writing letters, so each student promises to write letters to three of the others.*

(a) *Is it possible to arrange for each student to write three letters and also to receive three letters?*

There are several ways to do this; we can represent this as a graph with students represented by vertices and letter-writing represented as edges (actually, since correspondence is one-way, a better tool for this purpose is a *directed* graph, in which edges have orientation, but we can make do with undirected edges). We want a graph in which each vertex has degree 6 (to represent sending 3 letters and receiving 3). The easiest way to do this is by numbering our vertices $v_1,\ldots,v_9$ and connecting each v_i to v_{i+1}, v_{i+2}, and v_{i+3}, wrapping around on the addition so that v_8, for instance, is conencted to v_9, v_1, and v_2. Then, we have outbound connections from each vertex to its three successors, and inbound from its three predecessors in our arbitrary ordering.

(b) *Is it possible to arrange for each student to write letters to three other students and also to receive letters from the same three students?*

Let us represent this as a graph G in which the vertices are students and edges represent two-way communication; that is, writing to another and being written to by them.

In such a scheme, we want each vertex to have degree 3, since each student is in communication with 3 other students. Thus, $2\|G\| = \sum_{v\in G} \deg v = 9\cdot 3 = 27$. We thus have the surprising and contrary result that G must have $\frac{27}{2}$ edges. Since no graph has a nonintegral number of edges, no such graph G can possibly be constructed and the situation described is impossible.

5.1.26. *Show that in any group of 10 people there are either four mutual friends or three mutual strangers.*

Representing people as vertices of a graph, and acquaintance of two people as an edge between their associated nodes, this question is equivalent to the claim that for any 10-vertex graph G, there are either 4 mutually adjacent points (a K_4 subgraph of G), or 3 mutually non-adjacent points (a K_3 subgraph of G^c). We will start with a simple (and useful!) lemma:

Lemma 1. *If* $|G| \geq 6$*, then either* K_3 *is a subgraph of* G *or* K_3 *is a subgraph of* G^c*.*

Proof. Pick an arbitrary vertex v in G. There are two cases to be dealt with:

Case I: $\deg v \geq 3$**.** Thus, v has at least three neighbors, which we shall call u_1, u_2, and u_3. Since $v \sim u_1$ and $v \sim u_2$, a K_3 would be formed among v, u_1, and u_2 if $u_1 \sim u_2$. Similar arguments will guarantee the existence of a K_3 if $u_2 \sim u_3$ or $u_1 \sim u_3$. Thus, to prevent the appearance of a K_3 in v, we would require that $u_1 \nsim u_2$, $u_2 \nsim u_3$, and $u_1 \nsim u_3$. But then, these three vertices will be mutually non-adjacent, and thus form a K_3 in G^c.

Case II: $\deg v \leq 2$**.** Then, $\deg v \geq (5-2) = 3$ in G^c, so the argument presented in Case I, applied to vertex v in G^c instead of in G, proves that there is a K_3 in either G^c or $(G^c)^c$, which is just G. □

Now we can prove our main result:

Theorem 1. *If* $|G| \geq 10$*, then either* K_4 *is a subgraph of* G *or* K_3 *is a subgraph of* G^c*.*

Proof. Pick an arbitrary vertex v in G. There are two cases to be dealt with:

Case I: $\deg v \geq 6$**.** Thus, v has at least six neighbors. By the lemma above, among these six vertices there must be either three mutually non-adjacent vertices or three mutually adjacent vertices. If there are three mutually non-adjacent vertices, one of the structures we sought is present (a K_3 in G^c); if, on the other hand, there are three mutually adjacent vertices among these six vertices, let us call them u_1, u_2, and u_3. By mutual adjacency, $u_1 \sim u_2$, $u_1 \sim u_3$, and $u_2 \sim u_3$. Since all three are neighbors of v, it is also the case that $v \sim u_1$, $v \sim u_2$, and $v \sim u_3$. Thus, since all 6 possible edges among v, u_1, u_2, and u_3 exist, these vertices form a K_4 subgraph of G.

Case II: $\deg v \leq 5$**.** Then there are at least $9 - 5 = 4$ vertices to which v is not adjacent: call them u_1, u_2, u_3, and u_4. If any $u_i \nsim u_j$, then v, u_i, and u_j would be mutually nonadjacent (forming a K_3 subgraph of G^c), if, on the other hand every $u_i \sim u_j$, then u_1, u_2, u_3, and u_4 would be mutually adjacent, forming a K_4 subgraph of G. □

This result can be slightly improved to hold whenever $|G| \geq 9$ by combining the above argument with certain impossibility results on the parity of vertex degrees (see 5.1.13(b), 5.1.20(b)). It is *not* true when $|G| = 8$, and specific 8-vertex graphs can be produced containing neither a K_4 nor three mutually non-adjacent vertices.

5.2.2. *For any vertex v of a simple graph G with $\delta^- \geq k$, show that G has a path of length k with initial vertex v.*

Let us give v the alternative name u_0, and let us choose a neighbor of u_0 arbitrarily and call it u_1. Then, since $\deg u_1 \geq k$, u_1 is incident to at least k edges, at least $k-1$ of which do not go to u_0. Pick an arbitrary such edge, and label its other endpoint u_2, so that we now have a chain of distinct adjacent vertices u_0, u_1, u_2. Since $\deg u_2 \geq k$, u_2 is incident to at least $k-2$ edges which do not go to u_0 or u_1; we choose one and label the vertex at the other end u_3. We continue in like manner: for each vertex u_i, we know it is incident on k edges, and there are i endpoints we wish to avoid (u_0, u_1, u_2, $\ldots$, u_{i-1}). Thus, of these k edges, $k-i$ of them are acceptable, so we pick one of those arbitrarily and label its endpoint u_{i+1}. This process will yield a sequence of adjacent distinct vertices $u_0, u_1, \ldots, u_k$; it cannot be extended any further by these means since u_k's k neighbors might very well be exactly those vertices already in the path. However, we have successfully produced a length-k path by these means.

This argument may also be phrased by induction on k: the inductive hypothesis guarantees the existence of $u_0, u_1, \ldots, u_{k-1}$, and all that remains is to show that a single-node extension to some point u_k is possible.

5.2.4. *If a graph G has exactly two components that are both complete graphs with k and $p-k$ vertices respectively ($1 \leq k \leq p-1$) then:*

A complete graph on k nodes has $\binom{k}{2} = \frac{k^2-k}{2}$ edges; a complete graph on $p-k$ nodes has $\binom{p-k}{2} = \frac{k^2-2pk+p^2-p+k}{2}$ edges. Together the two components thus have $\binom{k}{2} + \binom{p-k}{2} = \frac{2k^2-2pk+p^2-p}{2}$ edges.

(a) *For which values of k is the number of edges a minimum?*

We can treat this as a good old calculus or algebra minimization problem. We want to minimize $\frac{2k^2-2pk+p^2-p}{2}$ where p is a constant and k a variable in $[1, p-1]$. Quick inspection of the quadratic rveals that it's a concave-upwards parabola with axis $k = \frac{1}{2}p$. Thus, the maximizing value of this continuous function is $\frac{p}{2}$; since k is actually an integer, we have $k = \lceil \frac{p}{2} \rceil$ and $k = \lfloor \frac{p}{2} \rfloor$ as the points closest to minimal as possible. The graph so produced has $\binom{\lceil \frac{p}{2} \rceil}{2} + \binom{\lfloor \frac{p}{2} \rfloor}{2}$ edges.

(b) *For which values of k is the number of edges a maximum?*

As above, this is the maximum vsalue on an interval for an upwards-concave parabola. Since such a function's values increase with distance from the vertex of the parabola, one of the endpoints of our domain will maximize the function $\frac{2k^2-2pk+p^2-p}{2}$. In fact, both of them do: both $k = 1$ and $k = p-1$ yield a graph with $\binom{p-1}{2}$ edges.

5.2.10. *Show that any two longest paths in a connected graph have at least one vertex in common.*

Suppose we have two vertex-disjoint paths of length n. We shall show that a longer path can be constructed, which is equivalent to stating that two longest paths cannot be vertex-disjoint.

Let our two paths of length n be denoted $u_0 \sim u_1 \sim u_2 \sim \cdots \sim u_n$ and $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_n$, and we operate under the assumption that all of these vertices are distinct. Now, let us consider a path from u_0 to v_0 given by $u_0 = w_0 \sim w_1 \sim w_2 \sim \cdots \sim w_k = v_0$. The vertices in this path are obviously *not* distinct from the u_i and v_i; in particular, we know at least one u_i occurs and at least one v_i occurs. Thus, considering the subsequence of the path $\{w_i\}$ consisting only of vertices from the two other paths, we have a sequence consisting of at least one u_i and at least one v_i. Thus, somewhere in this sequence is a pair of consecutive terms u_i, v_j. Extrapolating from this to the path of which it is a subsequence, we know that there is some subpath $u_i = w_t \sim w_{t+1} \sim w_{t+\ell} = v_j$ in which none of the vertices except the first and last are in either of the paths $\{u_i\}$ and $\{v_j\}$. Now we have two possibilities:

Case I: $i \leq j$**.** We may construct the following path by splicing together disjoint pieces of paths we already know about:

$$v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_j = w_{t+\ell} \sim w_{t+\ell-1} \sim \cdots \sim w_t = u_i \sim u_{i+1} \sim \cdots \sim u_n$$

This path is built up of fragments of length j, ℓ and $n - i$ in order, so the path has length $j + \ell + (n - i) = n + \ell + (j - i) > n$.

Case II: $i > j$**.** We may construct the following path by splicing together disjoint pieces of paths we already know about:

$$u_0 \sim u_1 \sim u_2 \sim \cdots \sim u_i = w_t \sim w_{t+1} \sim \cdots \sim w_{t+\ell} = v_j \sim v_{j+1} \sim \cdots \sim v_n$$

This path is built up of fragments of length i, ℓ and $n - j$ in order, so the path has length $i + \ell + (n - j) = n + \ell + (i - j) > n$.

Thus, in either case we can construct a path longer than n, so disjoint paths $\{u_k\}$ and $\{v_k\}$ cannot be the longest paths in the graph. Thus, if we have two longest paths in the graph, they are not vertex-disjoint.

5.2.12. *Show that if G is disconnected, G^c is connected.*

Since G is disconnected, we can find vertices u and v in G such that no path exists between u and v. In particular, u and v are non-adjacent.

Let us consider an arbitrary third vertex w in G. If w were adjacent to both u and v, u and v would have a path between them. Thus, w is non-adjacent to at least one of u and v in G. Since non-adjacency becomes adjacency in the complement, we know that in G^c, u is adjacent to v and w is adjacent to at least one of u or v. Thus, w is connected to both u and v in G^c.

Since the above argument depended on no special properties of w, it is true for every vertex of G distinct from u and v. Thus, every vertex in G^c is connected to u and v; by transitivity of connectivity, this means every vertex in G^c is connected to every other vertex, so G^c is connected.