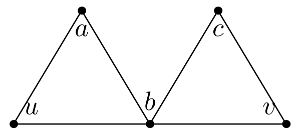
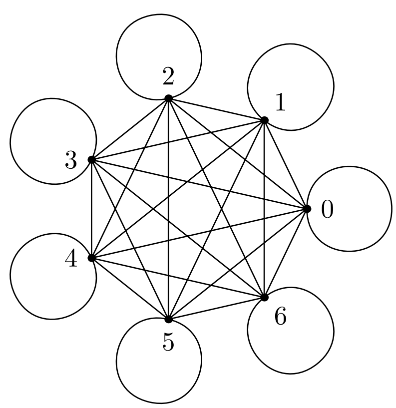
5.3.2. *Give an example of a graph G that has a circuit containing vertices u and v, but no cycle containing u and v.*

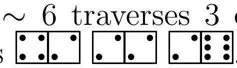


Here u and v are on the shared circuit $u \sim a \sim b \sim c \sim v \sim b \sim u$, but are on no shared cycle, since any circuit containing both u and v must visit the bottleneck vertex b twice and will thus not be a cycle.

5.3.6. *A domino is a 1×2 rectangular tile with between 0 and 6 dots on each half. In a complete set of 28 dominos, each combination of numbers appears exactly once on the two halves of some domino. Show that the dominos can be laid end to end with adjacent halves containing the same number of dots.*

Here, we may use an unorthodox representation: let halves of the dominoes be vertices, and the dominoes themselves be edges between their halves. Thus, we have 7 vertices, and every possible edge including loops, to account for doubles. The graph so produced is below:



Then, a walk on this graph represents a series of consecutive dominoes: for instance, the walk $4 \sim 2 \sim 2 \sim 6$ traverses 3 edges, representing 3 dominoes, and we have the series of dominoes . To visit each domino exactly once, we thus need to traverse each edge exactly once. That is to say, a sequence of all 28 dominoes satisfying the conditions given above is identical to an Eulerian walk on this graph. By inspection, every vertex of the above graph can be seen to have degree 8, so since every vertex is of degree 8, an Eulerian walk (or even an Eulerian cycle) can be constructed.

5.3.8. *If G is a graph that has all vertices of even degree, show that any edge e of G is contained in a cycle.*

We know that G has an Eulerian circuit, so there is clearly a circuit containing any particular edge e (the Eulerian circuit itself, which contains all edges). From here, the problem boils down to one stated in Exercise 5.3.1: if e is in a circuit, show that it must be in a cycle. Let us call the endpoints of e by the names u and v, so our Eulerian circuit includes the step $u \sim v$ representing its passage along e. Now let us consider the continuation of the circuit from here, along labeled points w_i: $u \sim v = w_0 \sim w_1 \sim \cdots \sim w_{k-1} \sim w_k = u$. If this circuit is a cycle, we're done. If not, then there must be a self-intersection $w_i = w_j$, in which case we simply drop the cycle-section between i and j and continue with the new circuit:

$$u \sim v = w_0 \sim w_1 \sim \cdots \sim w_{i-1} \sim w_i \sim w_j \sim w_{j+1} \sim \cdots \sim w_{k-1} \sim w_k = u$$

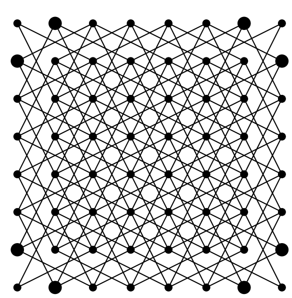
We repeat this procedure until we have a cycle, which is guaranteed to happen eventually, since the circuit only has finitely many vertices and hence only finitely many self-intersections.

5.3.10. (a) *A mythical chess piece is allowed to move one square vertically or one square horizontally. Show that it is not possible to move from one corner of an 8×8 chessboard and visit each square exactly once, ending at the diagonally opposite corner.*

This question is identical to finding a Hamiltonian path from the lower left corner to the upper right on a graph in which board positions are vertices on a graph and edges are valid moves. However, the problem can also be solved in a more intuitive way: each move is between squares of different colors, according to the usual coloring of the chessboard. We start in one corner and have 63 other squares to visit, so we must make 63 moves. After doing so, we will have changed color 63 times, so our final position must be a square of opposite color to the one we started on. However, opposite corners of the chess board are the same color.

(b) *A knight's move on a chessboard starts by first going two squares horizontally or vertically and ends by going one square perpendicularly. Is it possible to move a knight on an 8×8 chessboard so that it completes every move exactly once, where a move between two squares is said to be completed when it is made in either direction?*

This question is identical to finding an Eulerian path on a graph in which board positions are vertices on a graph and edges are valid knight's-moves. Such a graph is shown below:



While this graph is rather a mess, the most significant feature is the degree at each point (that is, the number of valid knight's-moves from that point. In the middle of the board, where the knight has free motion, there are 6 valid moves, and these points have degree of 6; points in the corner, on the other hand, only admit two possible moves and thus these points can be seen to have degree of 2. In particular, as pertains to Eulerian paths, we are particularly curious as to whether there are points of odd degree, and one can indeed find several: the 8 points one step away from the corners of the board (emphasized in the graph above with slightly heavier-weight dots), for instance, have degree of 3. Since there are more than two vertices of odd degree, an Eulerian path through this graph is not possible.

5.3.12. *Two cycles in a graph are said to be the same if they consist of the same cyclic rotation of vertices, whether or not they start at the same vertex.*

(a) *Count the number of different Hamiltonian cycles in the complete graph K_n.*

Let us label the vertices of K_n as u_1, u_2, u_3, ..., u_n. Since a Hamiltonian cycle visits every vertex of the graph, we know it must visit u_1; let us arbitrarily define u_1 as the "start" vertex, so as to not count the same cycle starting in different places multiple times. Since we have complete freedom of movement, a Hamiltonian cycle is simply an ordering of the vertices $u_1 \sim u_{i_2} \sim u_{i_3} \sim u_{i_4} \sim \cdots \sim u_{i_n} \sim u_1$. Every arrangement of $2, 3, 4, \ldots, n$ into $i_2, i_3, i_4, \ldots, i_n$ can be assembled into a path, since every possible edge is present in the complete graph. Thus, we have $(n-1)!$ different paths arising therefrom. The statement of this problem is vague on whether reversals of the vertex order is also considered identical, e.g. whether $u_1 \sim u_2 \sim u_3 \sim \cdots \sim u_n \sim u_1$ is the same path as $u_1 \sim u_n \sim u_{n-1} \sim u_{n-2} \sim \cdots \sim u_2 \sim u_1$; if you hold to this viewpoint, then we have actually counted every cycle twice with the above construction, so we would have $\frac{(n-1)!}{2}$ cycles instead.

(b) *Count the number of different Hamiltonian cycles in the complete bipartite graph $K_{n,n}$.*

As above, let us name the vertices in the two parts, with distinctive names for each part, for instance, $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. As above, we know a Hamiltonian cycle visits u_1, so we can fix u_1 as the start vertex. In this case we know a Hamiltonian cycle consists of distinct vertices picked alternatingly from the u_i and the v_i; e.g. $u_1 \sim u_2$ would be an inappropriate step in a Hamiltonian cycle because edges only go between the parts, not among a single part. Thus, a prototypical Hamiltonian cycle in this graph would be

$$u_1 \sim v_{i_1} \sim u_{j_2} \sim v_{i_2} \sim u_{j_3} \sim \cdots \sim u_{j_{n-1}} \sim v_{j_{n-1}} \sim u_{i_n} \sim v_{j_n} \sim u_1$$

where $i_1, \ldots, i_n$ can be any ordering of $1, \ldots n$, and $j_2, \ldots, j_n$ can be any ordering of $2, \ldots n$. These orderings can be done in $n!$ and $(n-1)!$ ways respectively, so there are $n!(n-1)!$ ways in total to select the indices on both u and v. If we consider reflections as identical, as commented on in the previous section, we divide by 2 to get $\frac{n!(n-1)!}{2}$.

5.3.16. *Show that if G is a simple graph with $\delta^- \geq 2$, then G contains a cycle of length at least $\delta^- + 1$.*

5.3.22. *Consider the n-cube Q_n, which has one vertex for every binary sequence of length n, two binary sequences being adjacent if and only if they differ in exactly one digit.*

(a) *Show that Q_n has a Hamiltonian cycle for every $n \geq 2$.*

We can actually do this recursively: we take a Hamiltonian path in Q_{n-1}, add an unchanging binary digit to it, and then change the unchanged digit and reverse the Hamiltonian path. This is possible by and large because Q_n can be represented as a pair of Q_{n-1}s with identical vertices attached to each other, so a Hamiltonian traversal of a Q_n is a matter of cleverly grafting together two Hamiltonian traversals of Q_{n-1}.

By way of illustration, start with a simple Hamiltonian path on Q_2: the path $00 \sim 01 \sim 11 \sim 10$. Now, to build a Hamiltonian cycle in Q_3, we would augment those by the bit "0" as such: $000 \sim 010 \sim 110 \sim 100$, traverse the edge $100 \sim 101$, and then reverse the given Q_2 Hamiltonian path, augmented with the bit "1": $101 \sim 111 \sim 011 \sim 001$, and finally, return to the start position, so our Hamiltonia cycle is:

$$000 \sim 010 \sim 110 \sim 100 \sim 101 \sim 111 \sim 011 \sim 001 \sim 000$$

Without the final step this is a Hamiltonian path, suitable for constructing a cycle on Q_4 by the above algorithm of duplication, augmentation, and reversal.

Hamiltonian paths on Q_n are generally known as *Gray codes*, and are used in designing test sequences for electromechanical switches and data buses.

(b) *For which values of n is Q_n Eulerian?*

It is clear that every vertex of Q_n has degree n, since an $n-$digit bit sequence differs by one bit from n different sequences: one resulting from toggling the first bit, one resulting from toggling the second, one resulting from toggling the third, etc. Since the criterion for a graph to be Eulerian is that its degree at every vertex is even, Q_n is Eulerian if and only if n is even.