1. **(8 points)** *A sample of phlebotinum-75 decays over time, losing 23% of its mass per year. We have just obtained a 20-gram sample of this element.*

   (a) **(3 points)** *Create a function $f(t)$ to describe the expected mass of phlebotinum-75 remaining in the sample t years from now.*

   Since the sample decays by 23% per year, the mass after a year is 77% of the former mass; thus, in t years, the sample has a mass equal to 0.77^t of the original mass. Thus, $f(t) = 20\left(0.77^t\right)$.

   (b) **(5 points)** *The sample is too small to be of further use to us after only 5 grams remain. How long will it take for this to happen?*

   We solve for t when $f(t) = 20$:

$$\begin{aligned} 20\left(0.77^t\right) &= 5 \\ 0.77^t &= \frac{5}{20} = \frac{1}{4} \\ t &= \log_{0.77}\frac{1}{4} = \frac{\ln\frac{1}{4}}{\ln 0.77} \approx 5.3 \text{ years} \end{aligned}$$

2. **(8 points)** *Given the function $f(x) = \frac{(x^2+1)(x+1)}{(x-2)(x+2)(2x+3))}$, answer the following questions preparatory to sketching the functions.*

   (a) **(2 points)** *What is the domain of the function?*

   The denominator is zero when $x-2=0$, $x+2=0$, or $2x+3=0$. These occur at $x=2$, $x=-2$, and $x=\frac{-3}{2}$ respectively. This prevents the function from being evaluatable at these points. The domain may be given as restriction on x in the form $x \neq 2, -2, \frac{-3}{2}$, or as the interval notation $(-\infty,-2)\cup(-2,\frac{-3}{2})\cup(\frac{-3}{2},2)\cup(2,\infty)$.

   (b) **(2 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large or very negative values of x, $g(x)$ is approximately equal to the quotient of the highest-degree terms in the numerator and denominator. Multiplying out the highest-degree terms in each factor yields $\frac{x^3}{3x^2} = \frac{1}{2}$, so over the long term $g(x)$ tends towards $\frac{1}{2}$. Thus, as $x \to \pm\infty$, $g(x) \to \frac{1}{2}$.

   (c) **(2 points)** *What are all the vertical asymptotes of the function?*

   We already found the zeroes of the denominator in the first section of this problem; since the numerator is nonzero at all of these, they are all asymptotes, so the function has vertical asymptotes at $x = 2, -2, \frac{-3}{2}$.

   (d) **(2 points)** *What are all the zeroes of the function?*

   The function is zero when the numerator is zero and the denominator is not zero. Of the two factors in the numerator, x^2+1 is never zero, since $x^2+1 \geq 1$, but $x+1$ is zero when $x=-1$. Noting that the denominator is nonzero at $x=-1$, we can conclude that $x=-1$ is a zero of this function.

3. **(8 points)** *Let* $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 1 \\ \sqrt{x+a} & \text{if } 1 < x \leq 6 \\ bx & \text{if } x > 6 \end{cases}$.

   What choices of a and b will make this function continuous?

Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 1$ and $x = 6$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 1$:

$$\begin{aligned} \lim_{x\to 1-} f(x) &= \lim_{x\to 1+} f(x) \\ \lim_{x\to 1-} x^2+1 &= \lim_{x\to 1+} \sqrt{x+a} \\ 1^2+1 &= \sqrt{1+a} \\ 2^2 &= 1+a \\ a &= 2^2-1 = 3 \end{aligned}$$

And likewise for $x = 6$:

$$\begin{aligned} \lim_{x\to 6-} f(x) &= \lim_{x\to 6+} f(x) \\ \lim_{x\to 6-} \sqrt{x+a} &= \lim_{x\to 6+} bx \\ \sqrt{6+3} &= 6b \\ 3 &= 6b \\ \frac{1}{2} &= b \end{aligned}$$

So our solution is to choose $a = \frac{8}{3}$ and $b = \frac{1}{2}$.

4. **(8 points)** *Let* $g(u) = \frac{2u^2-3u-5}{u+1}$.

   (a) **(1 point)** *Find* $\lim_{u\to -1} g(u)$.

   Note that $g(u) = \frac{(2u-5)(u+1)}{u+1}$. Thus, except at the point $u = -1$, $f(t) = 2u - 5$. Since the limit concerns the behavior not *at* $u = -1$ but in its vicinity, $\lim_{u\to -1} \frac{2u^2-3u-5}{u+1} = \lim_{u\to -1} 2u - 5 = -7$.

   (b) **(4 points)** *Using epsilon-delta methods, justify your result above.*

   Given a value of ϵ, we constrain $g(u)$ to be within ϵ of -7, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned} \left|\frac{2u^2-3u-5}{u+1} + 7\right| &< \epsilon \\ \left|\frac{(2u-5)(u+1)}{u+1} + 7\right| &< \epsilon \\ |2u-5+7| &< \epsilon \text{ and } u \neq -1 \\ |2u+2| &< \epsilon \text{ and } u \neq -1 \\ |u+1| &< \frac{\epsilon}{2} \text{ and } u \neq -1 \end{aligned}$$

   So, since it is sufficient to require x within $\frac{\epsilon}{2}$ of -1, we may establish δ to be $\frac{\epsilon}{2}$.

(c) **(3 points)** *State the mathematical definition of the statement* $\lim_{x\to+\infty} f(x) = L$.

Several statements of this are possible, in varying levels of opaqueness. Two are given below:

> Given a (presumably very small) value $\epsilon > 0$, we can find a (presumably very large) value of D such that if $x > D$, then $|f(x) - L| < \epsilon$.
> For any value $\epsilon > 0$, there is a D, such that if $x > D$, then $L - \epsilon \leq f(x) \leq L + \epsilon$.

Essentially, it is necessary to include in the definition that choice of D is a response to choice of ϵ, and that the boundary on D forcing x to be large ensures that $f(x)$ is within ϵ of L.

5. **(8 points)** *Let* $g(x) = -3x^2 + 7x - 2$.

(a) **(6 points)** *Using the difference quotient, find* $g'(x)$.

$$\begin{aligned}
g'(x) &= \lim_{h\to0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to0} \frac{[-3(x+h)^2 + 7(x+h) - 2] - (-3x^2 + 7x - 2)}{h} \\
&= \lim_{h\to0} \frac{-3x^2 - 6xh - 3h^2 + 7x + 7h - 2 - (-3x^2 + 7x - 2)}{h} \\
&= \lim_{h\to0} \frac{-6xh - 3h^2 + 7h}{h} \\
&= \lim_{h\to0} -6x - 3h + 7 \text{ justified since } h \neq 0 \\
&= -6x + 7
\end{aligned}$$

(b) **(2 points)** *Find the equation of the tangent line to* $g(x)$ *at the point* $(2, 0)$.

We know this line must pass through $(2, 0)$, with slope $g'(2) = -6 \cdot 2 + 7 = -5$. Using point-slope form, we get

$$(y - 0) = -5(x - 2)$$

which can also be expressed in slope-intercept form as $y = -5x + 10$.

6. **(8 points)** *Answer the following questions for the function* $f(t) = -2\cos(6\pi t) + 3$.

(a) **(3 points)** *What are its amplitude and period?*

Since the range, which is the distance from crests to troughs, has length 4, the amplitude, which is the distance from the centerline to either crests or troughs, is half of 4, or 2.

The period is the crest-to-crest distance. On an unmodified cosine function, this would be 2π. This cosine function has been compressed by a factor of 6π, so the period length has been compressed to $\frac{2\pi}{6\pi} = \frac{1}{3}$.

(b) **(3 points)** *What are its domain and range?*

There are no domain-interrupting expressions in the definition if $f(t)$, so the domain is $(-\infty, \infty)$, or alternatively, all real t.

The cosine function itself has range $[-1, 1]$, so multiplying by -2 flips it and stretches it to $[-2, 2]$; adding 3 shifts it to $[1, 5]$.

(c) **(2 points)** *Is it odd, even, both, or neither? Briefly justify your answer.*

The cosine finction itself is even, and has thus reflection symmetry over the y-axis. None of the actions taken (stretching horizontally, stretching vertically, or shifting vertically) affect this symmetry, so $f(t)$ is also even.

7. **(8 points)** *Evaluate the following limits; when a limit can not be evaluated, explain why or describe its behavior.*

(a) **(2 points)** $\lim_{t\to+\infty} \frac{2t^3-4t^2+7}{-t^4+5t^2-2}$.

In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so $\lim_{t\to+\infty} \frac{2t^3-4t^2+7}{-t^4+5t^2-2} = \lim_{t\to+\infty} \frac{2t^3}{-t^4} = \lim_{t\to+\infty} \frac{-2}{t} = 0$.

(b) **(2 points)** $\lim_{\theta\to\pi^-} \sin\theta$.

Since the sine function is continuous throughout, $\lim_{\theta\to\pi^-} \sin\theta = \sin\pi = 0$.

(c) **(2 points)** $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$.

Since we look near, but not at, $x = 1$, we can justify the cancellation $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} x+2 = 5$.

(d) **(2 points)** $\lim_{r\to 3} \frac{r^3-1}{r-3}$.

This rational function has a zero denominator but not a zero numerator at $r = 3$, so it has an infinite discontinuity there and thus the limit is unevaluatable.