1. **(12 points)** *Estimate the following values using appropriate linear approximations.*

   (a) **(6 points)** *Find a decimal approximation to* $(1.03)^7$.

   A good point at which to take our linearization of the seventh-power function is the nearby value 1; if $f(x) = x^7$, then $f(1) = 1$ and $f'(1) = 7 \cdot 1^6 = 7$, so the linearization of the seventh-power function around the point $a = 1$ is

$$f(x) \approx 1 + 7(x - 1)$$

   which, evaluated at $x = 1.03$, gives $f(1.03) \approx 1 + 7(0.03) = 1.21$. This isn't actually all that accurate, having an error of nearly 0.02.

   (b) **(6 points)** *Find a rational number approximately equal to* $\sqrt[3]{7.95}$.

   A good point at which to take our linearization of the cube-root function is the nearby value 8; if $f(x) = \sqrt[3]{x}$, then $f(8) = 2$ and $f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12}$, so the linearization of the cube-root function around the point $a = 8$ is

$$f(x) \approx 2 + \frac{1}{12}(x - 8)$$

   which, evaluated at $x = 7.95$, gives $f(83) \approx 2 - \frac{(}{0}.05)12 = \frac{479}{240}$. The error on this result is actually less than 0.000009.

2. **(12 points)** *Answer the following questions:*

   (a) **(6 points)** *Determine a region whose area is* $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n}\right) \ln\left(2 + \frac{5i}{n}\right)$.

   The area of a region under a curve $f(x)$ from $x = a$ to $x = b$ is known to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{b-a}{n} i\right),$$

   which bears a similarity except in details to the expression given. We must therefore find expressions to stand in for a, b, and $f(x)$ to make these expressions equivalent. Taking the most naïve decomposition of these equivalent expressions, we find that

$$\frac{b-a}{n} = \frac{5}{n}$$
$$f\left(a + \frac{b-a}{n} i\right) = \ln 2 + \frac{5}{n} i$$

   The first equation clearly establishes that $b - a = 5$; substituting this knowledge into the second equation, we find that our correspondences become:

$$b - a = 5$$
$$f\left(a + \frac{5}{n} i\right) = \ln 2 + \frac{5}{n} i$$

which lends itself to the obvious intepretation $f(x) = \ln x$ and $a = 2$ (other interpretations are possible, and will give rise to slightly different but equally correct answers). Then, since $b - a = 5$, $b = 5 + a = 7$. Thus, the expression we were given is the area-under-a-curve formula with $a = 2$, $b = 7$, and $f(x) = \ln x$, so this expression is the area of the region under the curve $y = \ln x$ between $x = 2$ and $x = 7$, which might be written as $\int_2^7 \ln x dx$.

(b) **(6 points)** *Find the general antiderivative of the function* $g(x) = 4 + \sqrt{x} - \sec x \tan x + \frac{2}{1+x^2}$.

We intepret this expression as $g(x) = 4x^0 + x^{1/2} - \sec x \tan x + \frac{2}{1+x^2}$. Using known antiderivative rules, we know that antiderivatives for x^0, $x^{1/2}$, $\sec x \tan x$, and $\frac{1}{1+x^2}$ are $\frac{x^1}{1}$, $\frac{x^{3/2}}{3/2}$ $\sec x$, and $\arctan x$ respectively, so the general antiderivative of $f(x)$ is

$$4\frac{x^1}{1} + \frac{x^{3/2}}{3/2} - \sec x + 2\arctan x + C = 4x + \frac{2x^{3/2}}{3} - \sec x + 2\arctan x + C$$

3. **(12 points)** *Answer the following questions related to the shape of the graph of* $f(x) = x^3 + 3x^2 - 24x + 6$.

(a) **(4 points)** *Where is it increasing? Where is it decreasing?*

$f'(x) = 3x^2+6x-24 = 3(x^2+2x-8) = 3(x+4)(x-2)$; multiplying constituent parts and noting their sign changes, we see that $f'(x)$ is positive (since both factors are negative) if $x < -4$, negative if $-4 < x < 2$, and positive if $x > 2$. Thus $f(x)$ is increasing when $x < -4$ or $x > 2$, and decreasing when $-4 < x < 2$ (some definitions of increase and decrease include the $f'(x) = 0$ case, so these intervals may include their endpoints, if desired).

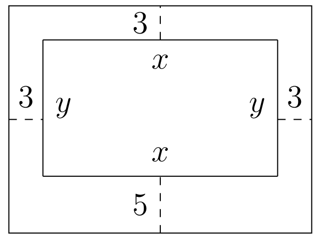
(b) **(4 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

From the factorization above, it is clear that $f(x) = 0$ when $x = -4$ and $x = 2$. Since $f(x)$ increases up to -4 and decreases from it, $x = -4$ is clearly a local maximum; since it decreases to 2 and increases after, $x = 2$ is a minimum. This result can also be obtained via the second derivative test, but doing so when you already have the signs of the first derivative at hand is not terribly necessary.

(c) **(4 points)** *Where is it concave up? Where is it concave down? Does it have any points of inflection?*

$f''(x) = 6x + 6$, so $f''(x) > 0$ if $x > -1$, and $f''(x) < 0$ if $x < -1$. Thus, $f(x)$ is concave up when $x > -1$, concave down when $x < -1$, and has a point of inflection at $x = -1$.

4. **(12 points)** *You are planning a design for a 1200-square-foot rectangular swimming pool, with a rectangle of paving around the entire pool. Around three sides of the pool you want to have a 3-foot paved strip; on the fourth side you want to have a 5-foot strip. What dimensions for the pool will minimize the necessary total area of the pool and poolside paving?*



The above drawing is a representation of the scenario described; we assign the pool's dimensions the labels of x and y; then the dimensions of the poolside area can be seen to be $(x+6)$ and $(y+8)$.

Thus, our goal is to minimize $(x+6)(y+8)$, subject to the constraint that $xy = 1200$. We may re-express this constraint as $y = \frac{1200}{x}$, so that the expression we seek to minimize is, in a single variable, $f(x) = (x+6)\left(\frac{1200}{x} + 8\right) = 1200 + 8x + \frac{7200}{x} + 48$. The interval of acceptable values on x is $[0, \infty)$, since our pool cannot have negative width but has no bound on its upper length.

Since $f'(x) = 8 - \frac{7200}{x^2}$, we see that $f'(x)$ will have 3 critical points: one when $x = 0$, where $f'(x)$ is undefined, and two at the solutions to $8 - \frac{7200}{x^2} = 0$, which occur at $x = \pm\sqrt{900} = \pm 30$. Our options for maximizing choices of x are thus the 3 critical points 0, -30, and 30, together with the interval endpoints 0 and the limiting behavior as $x \to \infty$. -30 is outside the interval and may be rejected out of hand. Evaluating at the other three points, we see that $f(x)$ does not exist at $x = 0$, but that $\lim_{x\to 0^+} f(x) = +\infty$; $\lim_{x\to+\infty} f(x) = +\infty$, and $f(30) = 1248 + 8 \cdot 30 + \frac{7200}{30}$. This last value need not be completely calculated; it is finite, and thus a better minimum than the other two prospects. Thus, our area-minimizing choice of x is 30, which has an associated value of $y = \frac{1200}{30} = 40$, so our optimal dimensions are 30×40.

5. **(12 points)** *Answer the following questions about approximation:*

   (a) **(6 points)** *Note that $\sqrt[5]{33}$ is a zero of the function $f(x) = x^5 - 33$. Choose an integer value of x_0 which is close to $\sqrt[5]{33}$. Use one step of Newton's method to develop x_1, a better rational approximation of $\sqrt[5]{33}$.*

   Note that $2 = \sqrt[5]{32}$, so 2 is quite close to $\sqrt[5]{33}$. Let $x_0 = 2$. Then, since $f(x) = x^5 - 33$ and $f'(x) = 5x^4$, we shall find that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^5 - 33}{5 \cdot 2^4} = 2 - \frac{-1}{80} = \frac{161}{80}$$

   This is off by only a little more than 0.00015.

   (b) **(6 points)** *Starting with an initial value of 1, use two iterations of Newton's method to approximate a zero of $f(x) = x^3 - 2x - 1$. Your answer need not be arithmetically simplified.*

   Let us start by observing that $f'(x) = 3x^2 - 2$. Using Newton's method once:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1^3 - 2 \cdot 1 - 1}{3 \cdot 1^2 - 2} = 1 - \frac{-2}{1} = 3$$

   And using it again:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{3^3 - 2 \cdot 3 - 1}{3 \cdot 3^2 - 2} = 3 - \frac{20}{25} = 2.2$$

This isn't actually all that close to the correct result of 1.6183, but a few more iterations would get it closer.

6. **(12 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

(a) **(3 points)** $\lim_{t\to+\infty} \frac{\ln t}{2t^2+1}$.

As t grows without bound, so do both $\ln t$ and $2t^2+1$, so this is a $\frac{\infty}{\infty}$ indeterminate form. Applying L'Hôpital's rule to this form gives

$$\lim_{t\to+\infty} \frac{\ln t}{2t^2+1} = \lim_{t\to+\infty} \frac{\frac{1}{t}}{4t} = \lim_{t\to+\infty} \frac{1}{4t^2}$$

in which the denominator grows without bound while the numerator, being a constant, does not. Thus, this expression approaches zero as t increases, so the limit is equal to zero.

(b) **(3 points)** $\lim_{\theta\to 0} \frac{\theta\cos\theta}{\sin\theta}$.

Evaluating the numerator and denominator of this expression with $\theta = 0$ gives zero for both, so we have a $\frac{0}{0}$ indeterminate form. Applying L'Hôpital's rule (and the product rule as necessary on the numerator) gives

$$\lim_{\theta\to 0} \frac{\theta\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{\cos\theta - \theta\sin\theta}{\cos\theta}$$

which can be directly evaluated to give $\frac{\cos 0 - 0\sin 0}{\cos 0} = \frac{1}{1} = 1$.

(c) **(3 points)** $\lim_{x\to+\infty} (x+4)e^{-x}$.

As x grows without bound, $x+4$ grows without bound also, but e^{-x} approaches zero. Thus, we have a $0 \times \infty$ indeterminate form here. This form must be reorganized into a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form before L'Hôpital's rule can be used: L'Hôpital only deals with those two specific forms. Fortunately, a quotient-reorganization is implied by the presence of e^{-x}, since $e^{-x} = \frac{1}{e^x}$, so we could reconsider this limit as $\lim_{x\to+\infty} \frac{x+4}{e^x}$ which is a $\frac{\infty}{\infty}$ indeterminate form and can be addressed with L'Hôpital's rule:

$$\lim_{x\to+\infty} \frac{x+4}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x}$$

and now the numerator is a constant while the denominator increases without bound, so the limit will be zero.

(d) **(3 points)** $\lim_{x\to 1} \frac{x^2-4x+4}{e^x}$.

This one can be solved by direct evaluation: $\lim_{x\to 1} \frac{x^2-4x+4}{e^x} = \frac{1^2-4\cdot 1+4}{e^1} = \frac{1}{e}$