1. **(8 points)** *Let* $f(x) = \begin{cases} ax & \text{if } x \leq 3 \\ 2^x & \text{if } 3 < x \leq 4 \\ x^2 + b & \text{if } x > 4 \end{cases}$.

   What choices of a and b will make this function continuous?

   Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 3$ and $x = 4$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 3$:

$$\begin{aligned} \lim_{x \to 3-} f(x) &= \lim_{x \to 3+} f(x) \\ \lim_{x \to 3-} ax &= \lim_{x \to 3+} 2^x \\ 3a &= 2^3 \\ a &= \frac{2^3}{3} = \frac{8}{3} \end{aligned}$$

   And likewise for $x = 4$:

$$\begin{aligned} \lim_{x \to 4-} f(x) &= \lim_{x \to 4+} f(x) \\ \lim_{x \to 4-} 2^x &= \lim_{x \to 4+} x^2 + b \\ 2^4 &= 4^2 + b \\ 16 &= 16 + b \\ 0 &= b \end{aligned}$$

   So our solution is to choose $a = \frac{8}{3}$ and $b = 0$.

2. **(8 points)** *Answer the following questions for the function* $f(t) = 4\sin(2\pi t) + 1$.

   (a) **(3 points)** *What are its domain and range?*

   There are no domain-interrupting expressions in the definition if $f(t)$, so the domain is $(-\infty, \infty)$, or alternatively, all real t.

   The sine function itself has range $[-1, 1]$, so multiplying by 4 stretches it to $[-4, 4]$; adding 1 shifts it to $[-3, 5]$.

   (b) **(2 points)** *Is it odd, even, both, or neither? Briefly justify your answer.*

   Even though the sine function is odd, the vertical shift ensures it is no longer symmetric about the origin, but is instead symmetric about $(0, 1)$. This makes it neither even nor odd, which a sketch of the graph could also show. Alternatively, it can be shown to be neither even nor odd by example: $f(\frac{1}{4}) = 5$, while $f(-\frac{1}{4}) = -3$, which conforms to neither the necessities for even or odd functions.

   (c) **(3 points)** *What are its amplitude and period?*

   Since the range, which is the distance from crests to troughs, has length 8, the amplitude, which is the distance from the centerline to either crests or troughs, is half of 8, or 4.

   The period is the crest-to-crest distance. On an unmodified sine function, this would be 2π. This sine function has been compressed by a factor of 2π, so the period length has been compressed to $\frac{2\pi}{2\pi} = 1$.

3. **(8 points)** *Let* $f(x) = 2x^2 - 3x$.

   (a) **(6 points)** *Using the difference quotient, find* $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[2(x+h)^2 - 3(x+h)] - (2x^2 - 3x)}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - (2x^2 - 3x)}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} \\ &= \lim_{h\to 0} 4x + 2h - 3 \text{ justified since } h \neq 0 \\ &= 4x - 3 \end{aligned}$$

   (b) **(2 points)** *Find the equation of the tangent line to* $f(x)$ *at the point* $(-1, 5)$.
   We know this line must pass through $(-1, 5)$, with slope $f'(-1) = 4(-1) - 3 = -7$. Using point-slope form, we get

$$(y - 5) = -7(x + 1)$$

   which can also be expressed in slope-intercept form as $y = -7x - 2$.

4. **(8 points)** *Evaluate the following limits; when a limit can not be evaluated, explain why or describe its behavior.*

   (a) **(2 points)** $\lim\limits_{u\to 1} \frac{(u^2-4u+3)}{(u-1)}$.

   Since we look near, but not at, $u = 1$, we can justify the cancellation $\lim_{u\to 1} \frac{(u^2-4u+3)}{(u-1)} = \lim_{u\to 1} \frac{(u-3)(u-1)}{(u-1)} = \lim_{u\to 1} u - 3 = -2$.

   (b) **(2 points)** $\lim\limits_{x\to+\infty} \frac{5x^3-2x+1}{4x^3+2x^2}$.

   In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so $\lim_{x\to+\infty} \frac{5x^3-2x+1}{4x^3+2x^2} = \lim_{x\to+\infty} \frac{5x^3}{4x^3} = \lim_{x\to+\infty} \frac{5}{4} = \frac{5}{4}$.

   (c) **(2 points)** $\lim\limits_{\theta\to\frac{\pi}{2}^+} 3\tan\theta$.

   $\tan\theta$ has an infinite discontinuity at $\theta = \frac{\pi}{2}$ (and at every odd multiple of $\frac{\pi}{2}$). Because there is an infinite discontinuity, this limit cannot be evaluated. We may, but need not, explicitly describe its behavior, which is that values of θ slightly above $\frac{\pi}{2}$ lead to extremely negative values of $3\tan\theta$ with the statement $\lim_{\theta\to\frac{\pi}{2}^+} 3\tan\theta = -\infty$.

   (d) **(2 points)** $\lim\limits_{t\to 2} \frac{t^2-4}{t+3}$.

   This rational function is defined at $t = 2$, so its limit is simply the evaluation $\frac{2^2-4}{2+3} = \frac{0}{5} = 0$.

5. **(8 points)** *Let* $f(t) = \frac{-3t^2+12}{t+2}$.

(a) **(1 point)** *Find* $\lim_{t\to -2} f(t)$.

Note that $f(t) = \frac{(-3t+6)(t+2)}{t+2}$. Thus, except at the point $t = -2$, $f(t) = -3t+6$. Since the limit concerns the behavior not *at* $t = -2$ but in its vicinity, $\lim_{t\to -2} \frac{-3t^2+12}{t+2} = \lim_{s\to t} -3t+6 = 12$.

(b) **(4 points)** *Using epsilon-delta methods, justify your result above.*

Given a value of ϵ, we constrain $f(t)$ to be within ϵ of 12, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned}
\left|\frac{-3t^2+12}{t+2} - 12\right| &< \epsilon \\
\left|\frac{(-3t+6)(t+2)}{t+2} - 12\right| &< \epsilon \\
|-3t+6-12| &< \epsilon \text{ and } t \neq -2 \\
|-3t-6| &< \epsilon \text{ and } t \neq -2 \\
\left|\frac{-3t-6}{-3}\right| &< \frac{\epsilon}{-3} \text{ and } t \neq -2 \\
|t+2| &< \frac{\epsilon}{3} \text{ and } t \neq -2 \\
0 < |t+2| &< \frac{\epsilon}{3}
\end{aligned}$$

So, since it is sufficient to require x within $\frac{\epsilon}{3}$ of 2, we may establish δ to be $\frac{\epsilon}{3}$.

(c) **(3 points)** *State the mathematical definition of the statement* $\lim_{x\to -\infty} f(x) = +\infty$.

Several statements of this are possible, in varying levels of opaqueness. Two are given below:

> Given a (presumably very large) value E, we can find a (presumably very negative) value of D such that if $x < D$, then $f(x) > E$.
> For any value E, there is a D, such that if $x < D$, then $f(x) > E$.

Essentially, it is necessary to include in the definition that choice of D is a response to choice of E, and that D acting as a limit on the value of x ensures that $f(x)$ is limited by E.

6. **(8 points)** *Given the function* $g(x) = \frac{(x-5)(x-2)}{(x^2+3)(x-2)(x+4)}$, *answer the following questions preparatory to sketching the functions.*

(a) **(2 points)** *What is the domain of the function?*

Note that x^2+3 is never zero, but that the denominator is zero when x is 2 or -4. This prevents the function from having a value at these points. The domain may be given as restriction on x in the form $x \neq 2, -4$, or as the interval notation $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$.

(b) **(2 points)** *What are all the zeroes of the function?*

The function is zero when the numerator is zero and the denominator is not zero (The $\frac{0}{0}$ form, as we have seen, is not necessarily a zero or an asymptote, and is furthermore outside the domain of our function). Of the two factors in the numerator, $(x-5)$ is zero when $x = 5$, and $(x-2)$ is zero when $x = 2$. Since 2 is also a zero of the denominator, the rational function as a whole has only $x = 5$ as a zero.

(c) **(2 points)** *What are all the vertical asymptotes of the function?*

We already found the zeroes of the denominator in the first section of this problem; however, as we saw in the second section, $x = 2$ is not an asymptote. Thus the only asymptote is the other undefined value of the function, at $x = -4$.

(d) **(2 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

For very large or very negative values of x, $g(x)$ is approximately equal to the quotient of the highest-degree terms in the numerator and denominator. Multiplying out the highest-degree terms in each factor yields $\frac{x^2}{x^4} = \frac{1}{x^2}$, so over the long term $g(x)$ tebnds towards zero, like the function $\frac{1}{x^2}$. Thus, as $x \to \pm\infty$, $g(x) \to 0$.

7. **(8 points)** *Transylvania Polygnostic University currently has 3000 students. Enrollment is expected to rise by 4% each year.*

(a) **(3 points)** *Create a function $f(t)$ to describe the expected number of students t years from now.*

Since enrollment grows by 4% per year, the enrollemnt after an hou is 104% of the former enrollment; thus, in t years, the university has a factor equal to 1.04^t of the original enrollemnt. Thus, $f(t) = 3000\,(1.04^t)$.

(b) **(5 points)** *How many years will it take for enrollment to reach 4000 students?*

We solve for t when $f(t) = 4000$:

$$\begin{aligned} 3000\left(0.82^t\right) &= 4000 \\ 1.04^t &= \frac{4000}{3000} = \frac{4}{3} \\ t &= \log_{1.04}\frac{4}{3} = \frac{\ln\frac{4}{3}}{\ln 1.04} \approx 7.3 \text{ years} \end{aligned}$$