1. **(8 points)** *Find an equation of the tangent line to the curve $y = \sqrt{25 - x^2}$ at $(3, 4)$.*

   Using the chain rule, if $y = \sqrt{u}$ and $u = 25 - x^2$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{u}}(-2x) = \frac{-x}{\sqrt{25-x^2}}$.

   Evaluated specifically at the point $x = 3$, one finds that $\frac{dy}{dx} = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$. Thus the slope of the tangent line to this curve at the point $(3, 4)$ is $\frac{-3}{4}$, and using point-slope form, the tangent line has equation

   $$y - 4 = \frac{-3}{4}(x - 3)$$

   which would, if written in slope-intercept form, be $y = \frac{-3}{4}x + \frac{25}{4}$.

2. **(8 points)** *Answer the following derivative-related questions.*

   (a) **(4 points)** *Find $\frac{d}{dx}\cos(e^{\sqrt[3]{x}})$.*

   This is a double chain-rule application. Let $u = e^{\sqrt[3]{x}}$, and $v = \sqrt[3]{x} = x^{1/3}$, so then:

   $$\begin{aligned}
   \frac{d}{dx}\cos u &= \frac{du}{dx}\frac{d}{du}\cos u \\
   &= \left(\frac{d}{dx}e^v\right)\left(\frac{d}{du}\cos u\right) \\
   &= \left(\frac{dv}{dx}\frac{d}{dv}e^v\right)\left(\frac{d}{du}\cos u\right) \\
   &= \left(\frac{d}{dx}\sqrt[3]{x}\right)\left(\frac{d}{dv}e^v\right)\left(\frac{d}{du}\cos u\right) \\
   &= \left(\frac{d}{dx}x^{1/3}\right)\left(\frac{d}{dv}e^v\right)\left(\frac{d}{du}\cos u\right) \\
   &= \left(\frac{1}{3}x^{-2/3}\right)(e^v)(-\sin u) \\
   &= -\frac{x^{-2/3}}{3}e^{\sqrt[3]{x}}\sin e^{\sqrt[3]{x}}
   \end{aligned}$$

   (b) **(2 points)** *If $f(x) = \frac{\sec x}{e^x}$, find $f'(x)$.*

   This is a straightforward quotient-rule application:

   $$f'(x) = \frac{e^x \frac{d}{dx}\sec x - \sec x \frac{d}{dx}e^x}{(e^x)^2} = \frac{e^x \sec x \tan x - e^x \sec x}{e^{2x}}$$

   This may, but need not, be simplified to $\frac{\sec x(\tan x - 1)}{e^x}$.

   (c) **(2 points)** *For $y = t\cot(t)$, find $\frac{dy}{dt}$.*

   This is a straightforward product-rule application:

   $$\frac{dy}{dt} = \left(\frac{d}{dt}t\right)\cot t + t\frac{d}{dt}\cot t = 1\cot t + t(-\csc^2 t)$$

   This may, but need not, be simplified to $\cot t - t\csc^2 t$.

3. **(8 points)** *Imre is twelve miles north of the Parliament, jogging southwards at six miles per hour; János is five miles to the east of Parliament, walking eastwards at three miles per hour.*

   (a) **(4 points)** *Is the distance between Imre and János increasing or decreasing, and at what rate?*

   Let us name various quantities which will change over time. First, we define time itself measured in hours as $t$, and name the physical quantities $x$, János's distance to the east of Parliament; $y$, Imre's distance to the north of Parliament; and $s$, the distance between them. Several relationships arise as a result of the problem statement: $x = 5$ at the time being investigated, $y = 12$ at the time being investigates, $\frac{dx}{dt} = 3$ (which is positive because János's distance from the Parliament is being increased by his movement), and $\frac{dy}{dt} = -6$ (which is negative because Imre's distance from the Parliament is being decreased by his movement). We also know that $s^2 = x^2 + y^2$ by the Pythagorean theorem, which will give us both the current distance between the two ($s = \sqrt{5^2 + 12^2} = 13$) and a relationship to be differentiated:

$$\begin{aligned}
\frac{d}{dt}s^2 &= \frac{d}{dt}(x^2 + y^2) \\
2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
\frac{ds}{dt} &= \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2s} \\
&= \frac{2 \cdot 5 \cdot 3 + 2 \cdot 12 \cdot (-6)}{2 \cdot 13} = \frac{-57}{13}
\end{aligned}$$

   Since this quantity is negative, the distance between the two is decreasing, at a rate of $\frac{57}{13}$ miles per hour.

   (b) **(4 points)** *In an hour, will the distance between Imre and János increasing or decreasing, and at what rate?*

   An hour later, the same relationships hold as above, but now $x = 5+3 = 8$, $y = 12-6 = 6$, and $s = \sqrt{8^2 + 6^2} = 10$, so we can calculate

$$\frac{ds}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2s} = \frac{2 \cdot 8 \cdot 3 + 2 \cdot 6 \cdot (-6)}{2 \cdot 10} = \frac{-6}{5}$$

   so they are still becoming closer together, but at the much slower rate of $\frac{6}{5}$ miles per hour.

4. **(8 points)** *Differentiate* $\frac{\arctan t}{\ln(\sin t)}$ *with respect to* $t$*.*

   This will require application of the quotient rule and the chain rule. We might anticipate needing to calculate $\frac{d}{dt}\ln(\sin t)$, and do so pre-emptively, or defer it until necessary. In either case, we would let $y = \ln u$, $u = \sin t$, and calculate $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dt} = \cos t$ to give the result from the chain rule that

$$\frac{d}{dt}\ln(\sin t) = \frac{dy}{du}\frac{du}{dt} = \frac{1}{u}\cos t = \frac{\cos t}{\sin t} = \cot t$$

Now we may invoke the quotient rule, armed with this knowledge:

$$\begin{aligned}\frac{d}{dt}\frac{\arctan t}{\ln(\sin t)} &= \frac{\ln(\sin t)\left(\frac{d}{dt}\arctan t\right) - \arctan t\left(\frac{d}{dt}\ln(\sin t)\right)}{(\ln(\sin t))^2} \\ &= \frac{\ln(\sin t)\left(\frac{1}{1+t^2}\right) - \arctan t\cot t}{(\ln(\sin t))^2}\end{aligned}$$

5. **(8 points)** *The* folium of Descartes *is a curve satisfying the equation* $x^3 + y^3 - 5xy = 0$.

   (a) **(6 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*

   Taking the derivative of each side, and using rules as necessary:

$$\begin{aligned}\frac{d}{dx}\left(x^3 + y^3 - 5xy\right) &= \frac{d}{dx}0 \\ 3x^2 + \frac{d}{dx}y^3 - 5\left(y + x\frac{dy}{dx}\right) &= 0 \\ 3x^2 + \frac{dy}{dx}\frac{d}{dy}y^3 - 5\left(y + x\frac{dy}{dx}\right) &= 0 \\ 3x^2 + 3y^2\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} &= 0 \\ (3y^2 - 5x)\frac{dy}{dx} &= 5y - 3x^2 \\ \frac{dy}{dx} &= \frac{5y - 3x^2}{3y^2 - 5x}\end{aligned}$$

   (b) **(2 points)** *Identify conditions on $x$ and $y$ for the tangent lines to the folium to be horizontal and vertical (label which is which!).*

   The tangent line is horizontal when $\frac{dy}{dx} = 0$, which is the case when its numerator is zero. Thus, the criterion for a tangent line to be horizontal is $5y - 3x^2 = 0$.

   The tangent line is vertical when $\frac{dy}{dx}$ is undefined due to an infinite asymptote. This will occur when the denominator of $\frac{dy}{dx}$ is zero. Thus, the criterion for a tangent line to be vertical is $3y^2 - 5x = 0$.

6. **(8 points)** *A collection of biological samples is taken from a −200°F deep-freeze into a 50°F lab. After 10 minutes it has warmed up to -150°F.*

   (a) **(4 points)** *Produce a function $T(t)$ modeling the samples' temperature $t$ minutes after they are brought into the lab.*

   We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 50°F, so our temperature model will be $T(t) = 50 + Ce^{kt}$; it remains only to find $C$ and $k$ to have a final model.

   Since the samples have a temperature of -200°F immediately upon removal from the freezer, $T(0) = -200$. Evaluating the left side of this equation, we find that $50 + Ce^0 = -200$; thus $C = -250$.

Since the samples have a temperature of -150°F ten minutes later, we know that $T(10) = -150$. Expanding $T(10)$, we find that:

$$\begin{aligned} 50 - 250e^{k\cdot 10} &= -150 \\ -250e^{10k} &= -200 \\ e^{10k} &= \frac{4}{5} \\ 10k &= \ln\frac{4}{5} \\ k &= \frac{\ln\frac{4}{5}}{10} \end{aligned}$$

Assembling this value of $k$ into our equation, we find that

$$T(t) = 50 - 250e^{\frac{\ln\frac{4}{5}}{10}t}$$

(b) **(2 points)** *The samples will become biologically active when they reach 0°F. How long will it take for this to occur?*

Since the samples become active when $T(t) = 0$, we want to find the value of $t$ satisfying that equation:

$$\begin{aligned} 50 - 250e^{\frac{\ln\frac{4}{5}}{10}t} &= 0 \\ -250e^{\frac{\ln\frac{4}{5}}{10}t} &= -50 \\ e^{\frac{\ln\frac{4}{5}}{10}t} &= \frac{1}{5} \\ \frac{\ln\frac{4}{5}}{10}t &= \ln\frac{1}{5} \\ t &= \frac{10\ln\frac{1}{5}}{\ln\frac{4}{5}} \approx 72 \text{ minutes} \end{aligned}$$

(c) **(2 points)** *How quickly are the samples' temperature changing ten minutes after being brought into the lab?*

The speed of the temperature change in ten minutes is $T'(10)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = -250\frac{\ln\frac{4}{5}}{10}e^{\frac{\ln\frac{4}{5}}{10}t}$$

so $T'(0) = -25(\ln\frac{4}{5})e^{\ln\frac{4}{5}} = -20\ln\frac{4}{5}$. This is approximately 4.46, signifying that the samples are warming (rising in temperature) by 4.46 degrees per minute.

7. **(8 points)** *Calculate* $\frac{d}{dx}\arcsin(x^2\tan x)$.

This is a chain rule problem, which will involve a product rule as a sub-problem. Letting $y = \arcsin u$ and $u = x^2\tan x$, we can calculate the derivatives $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and

$$\frac{du}{dx} = \left(\frac{d}{dx}x^2\right)\tan x + x^2\frac{d}{dx}\tan x = 2x\tan x + x^2\sec^2 x$$

so the original derivative is the product:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\left(2x\tan x + x^2\sec^2 x\right) = \frac{2x\tan x + x^2\sec^2 x}{\sqrt{1-(x^2\tan x)^2}}$$