1. **(12 points)** *Answer the following questions about approximation:*

   (a) **(6 points)** *Choose an integer value of x_0 approximating $\sqrt[3]{26}$ as well as possible. Use one step of Newton's method to develop a better rational approximation x_1.*

   We would here choose $x_0 = 3$, since $3 = \sqrt[3]{27}$, providing a plausible approximation for $\sqrt[3]{26}$. A simple function with $\sqrt[3]{26}$ as a zero is $f(x) = x^3 - 26$, so we will use Newton's method on this function (which has derivative $f'(x) = 3x^2$):

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3^3 - 26}{3 \cdot 3^2} = 3 - \frac{1}{27} = \frac{80}{27}$$

   This is actually a quite excellent approximation, with an error of less than 0.0005.

   (b) **(6 points)** *Starting with an initial value of 1, use two iterations of Newton's method to approximate a zero of $f(x) = x^4 - 5x + 1$. Your answer need not be arithmetically simplified.*

   Our initial value is 1, which we represent symbolically as $x_0 = 1$. The function of which we want a zero is $f(x) = x^4 - 5x + 1$, with derivative $f'(x) = 4x^3 - 5$. Using Newton's method on x_0, we find that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^4 - 5 \cdot 1 + 1}{4 \cdot 1^3 - 5} = 1 - \frac{-3}{-1} = -2$$

   And using Newton's method again, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2 - \frac{(-2)^4 - 5(-2) + 1}{4(-2)^3 - 5} = -2 - \frac{27}{-37} = \frac{-47}{37}$$

   which is actually not a very good approximation for the root at all, but Newton's method does not always find a root quickly.

2. **(12 points)** *Answer the following questions:*

   (a) **(6 points)** *Find the general antiderivative of the function $f(x) = 3x^3 + 4\sec^2 x - e^x + \frac{2}{x}$.*

   Using known antiderivative rules, we know that antiderivatives for x^3, $\sec^2 x$, e^x, and $\frac{1}{x}$ are $\frac{x^4}{4}$, $\tan x$, e^x, and $\ln|x|$ respectively, so the general antiderivative of $f(x)$ is

$$\frac{3}{4}x^4 + 4\tan x - e^x + 2\ln|x| + C$$

   (b) **(6 points)** *Determine a region whose area is $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right) \sqrt[3]{1 + \frac{2i}{n}}$.*

   The area of a region under a curve $f(x)$ from $x = a$ to $x = b$ is known to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{b-a}{n} i\right),$$

   which bears a similarity except in details to the expression given. We must therefore find expressions to stand in for a, b, and $f(x)$ to make these expressions equivalent. Taking the most naïve decomposition of these equivalent expressions, we find that

$$\frac{b-a}{n} = \frac{2}{n}$$
$$f\left(a + \frac{b-a}{n}i\right) = \sqrt[3]{1 + \frac{2}{n}i}$$

The first equation clearly establishes that $b - a = 2$; substituting this knowledge into the second equation, we find that our correspondences become:

$$b - a = 2$$
$$f\left(a + \frac{2}{n}i\right) = \sqrt[3]{1 + \frac{2}{n}i}$$

which lends itself to the obvious intepretation $f(x) = \sqrt[3]{x}$ and $a = 1$ (other interpretations are possible, and will give rise to slightly different but equally correct answers). Then, since $b - a = 2$, $b = 2 + a = 3$. Thus, the expression we were given is the area-under-a-curve formula with $a = 1$, $b = 3$, and $f(x) = \sqrt[3]{x}$, so this expression is the area of the region under the curve $y = \sqrt[3]{x}$ between $x = 1$ and $x = 3$, which might be written as $\int_1^3 \sqrt[3]{x}dx$.

3. **(12 points)** *Find approximations to the following values using appropriate linearizations.*

   (a) **(6 points)** *Find a rational number approximately equal to* $\sqrt{83}$.

   A good point at which to take our linearization of the square-root function is the nearby value 81; if $f(x) = \sqrt{x}$, then $f(81) = 9$ and $f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$, so the linearization of the square-root function around the point $a = 81$ is

   $$f(x) \approx 9 + \frac{1}{18}(x - 81)$$

   which, evaluated at $x = 83$, gives $f(83) \approx 9 + \frac{2}{18} = \frac{82}{9}$. This has an error of less than 0.001, which is not too bad.

   (b) **(6 points)** *Find an approximation to* $(1.998)^5$.

   A good point at which to take our linearization of the fifth-power function is the nearby value 2; if $f(x) = x^5$, then $f(2) = 32$ and $f'(2) = 5 \cdot 2^4 = 80$, so the linearization of the fifth-power function around the point $a = 2$ is

   $$f(x) \approx 2 + 80(x - 2)$$

   which, evaluated at $x = 1.998$, gives $f(1.998) \approx 32 + 80(-0.002) = 31.84$. This has an error of less than 0.0003.

4. **(12 points)** *You are designing a rectangular factory with 2400 square feet of floor space with walls on three sides and a loading dock on one side. Safety regulations demand that there be a 2-foot space between machinery and the walls, and a 4-foot space between machinery and the*

loading dock. What dimensions for the factory maximize the amount of space available for machinery?

Let us arbitrarily orient the factory so that the loading dock is on the top edge, and label the factory's horizontal and vertical dimensions as x and y respectively; thus the area of the factory itself is xy; the floor space usable for machinery will have dimensions $(x-4)\times(y-6)$, since the clearance on the left and right decrease the horizontal dimensions by $2+2$, while the 2-foot clearance on the bottom and 4-foot clearance on the top reduce the vertical space available by 6.

Thus, our goal is to maximize $(x-4)(y-6)$, subject to the constraint that $xy = 2400$. We may re-express this constraint as $y = \frac{2400}{x}$, so that the expression we seek to maximize is, in a single variable, $f(x) = (x-4)\left(\frac{2400}{x} - 6\right) = 2400 - 6x - \frac{9600}{x} + 24$. The interval of acceptable values on x is $[4, 400]$; in order to accomodate the necessary margins ($x \geq 4$, and $y \geq 6$).

Since $f'(x) = \frac{9600}{x^2} - 6$, we see that $f'(x)$ will have 3 critical points: one when $x = 0$, where $f'(x)$ is undefined, and two at the solutions to $\frac{9600}{x^2} - 6 = 0$, which occur at $x = \pm\sqrt{1600} = \pm 40$. Our options for maximizing choices of x are thus the 3 critical points 0, -40, and 40, together with the interval endpoints 4 and 400. 0 and -40 are outside the interval and may be rejected out of hand. Evaluating at the other three points, we see that $f(4) = (4-4)(600) = 0$; $f(400) = (400-4)(6-6) = 0$, and $f(40) = (40-4)(60-6) > 0$. Thus, the maximizing factory dimensions are 40×60.

5. **(12 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

   (a) **(3 points)** $\lim_{x\to 0^+} x^2 \csc x$.

   As x approaches zero from the right, x^2 approaches zero while $\csc x$ grows without bound, so this expression is the indeterminate form $0 \cdot \infty$. We may rephrase this as a $\frac{0}{0}$ indeterminate form by expressing it as $\frac{x^2}{\sin x}$. Applying L'Hôpital's rule to this form gives $\lim_{x\to 0^+} \frac{x^2}{\sin x} = \lim_{x\to 0^+} \frac{2x}{\cos x} = \frac{0}{1} = 0$.

   (b) **(3 points)** $\lim_{\theta\to\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}$.

   Evaluating the numerator and denominator of this expression gives $\frac{1}{2}$ and $\frac{3}{4}$ respectively; thus this is not an indeterminate form, and direct substitution suffices to give the answer $\frac{2}{3}$.

   (c) **(3 points)** $\lim_{t\to\infty} \frac{e^t}{2t+1}$.

   As t grows without bound, so do both e^t and $2t+1$, so this is a $\frac{\infty}{\infty}$ indeterminate form. Applying L'Hôpital's rule to this form gives

   $$\lim_{x\to\infty} \frac{e^t}{2t+1} = \lim_{x\to\infty} \frac{e^t}{2}$$

   in which the numerator grows without bound while the denominator, being a constant, does not. Thus, this expression grows withut bound as t grows without bound, so the limit does not exist (or, more specifically, $\lim_{x\to\infty} \frac{e^t}{2t+1} = +\infty$, which describes a particular type of limit-nonexistence).

   (d) **(3 points)** $\lim_{x\to -1} \frac{x^2+2x+1}{xe^x + e^{-1}}$.

Evaluating the numerator and denominator of this expression with $x = -1$ gives zero for both, so we have a $\frac{0}{0}$ indeterminate form. Applying L'Hôpital's rule gives

$$\lim_{x\to-1} \frac{x^2+2x+1}{xe^x+e^{-1}} = \lim_{x\to-1} \frac{2x+2}{xe^x+e^x}$$

which still evaluates to a $\frac{0}{0}$ form, so we apply L'Hôpital's rule again:

$$\lim_{x\to-1} \frac{x^2+2x+1}{xe^x+e^{-1}} = \lim_{x\to-1} \frac{2}{xe^x+2e^x}$$

And now we can evaluate to get $\frac{2}{-e^{-1}+2e^{-1}} = \frac{2}{e^{-1}} = 2e$

6. **(12 points)** *Answer the following questions related to the shape of the graph of* $f(x) = x^3 - 3x^2 - 45x + 2$.

   (a) **(4 points)** *Where is it increasing? Where is it decreasing?*
   $f'(x) = 3x^2 - 6x - 45 = 3(x^2 - 3x - 15) = 3(x+3)(x-5)$; multiplying constituent parts and noting their sign changes, we see that $f'(x)$ is positive (since both factors are negative) if $x < -3$, negative if $-3 < x < 5$, and positive if $x > 5$. Thus $f(x)$ is increasing when $x < -3$ or $x > 5$, and decreasing when $-3 < x < 5$ (some definitions of increase and decrease include the $f'(x) = 0$ case, so these intervals may include their endpoints, if desired).

   (b) **(4 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*
   From the factorization above, it is clear that $f(x) = 0$ when $x = -3$ and $x = 5$. Since $f(x)$ increases up to -3 and decreases from it, it is clearly a local maximum; since it decreases to 5 and increases after, $x = 1$ is a minimum. This result can also be obtained via the second derivative test, but doing so when you already have the signs of the first derivative at hand is not terribly necessary.

   (c) **(4 points)** *Where is it concave up? Where is it concave down? Does it have any points of inflection?*
   $f''(x) = 6x - 6$, so $f''(x) > 0$ if $x > 1$, and $f''(x) < 0$ if $x < 1$. Thus, $f(x)$ is concave up when $x > 1$, concave down when $x < 1$, and has a point of inflection at $x = 1$.

> Before writing this essay, I examined 85 separate and distinct calculus books. I looked at all of their prefaces, all of their applications of maxima and minima, and all of their treatments of L'Hospital's Rule. By the way, I found five different spellings of L'Hospital. There were the two you would expect [L'Hôpital and L'Hospital –DJW], and Lhospital, as L'Hospital sometimes spelled his name. In addition, one author, not wanting to take chances, had it L'Hôspital, and one thought it was Le Hospital.
> —Underwood Dudley, Review of *Calculus with Analytic Geometry*