1. **(12 points)** *Evaluate the following integrals:*

   (a) **(4 points)** $\int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x} dx$.
   We shall re-express $\frac{1}{\cos x}$ as $\sec x$, so that

   $$\int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x} dx = \int_{\pi/6}^{\pi/3} \sec^2 x dx = \tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}}$$

   (b) **(4 points)** $\int \frac{1}{x} \sec^2(\ln x) dx$.
   Since we have $\sec^2 \ln x$ in the integrand, it makes sense to use the substitution $u = \ln x$ to simplify this expression to $\sec^2 u$. We are particularly fortunate since then $du = \frac{1}{x} dx$, and a division by x is in fact present in our integral. We may now perform the u-substitution:

   $$\int \frac{1}{x} \sec^2(\ln x) dx = \int \sec^2 u du = \tan u + C = \tan \ln x + C$$

   (c) **(4 points)** $\int_0^1 x^3 \sin(x^4 + 2) dx$.
   Since we have $\sin(x^4+2)$ in the integrand, it makes sense to use the substitution $u = x^4+2$ to simplify this expression to $\sin u$. We are particularly fortunate since then $du = 4x^3 dx$, and a multiplication by x^3 is in fact present in our integral. We note also change in the limits; when $x = 0$, $x^4 + 2 = 2$, and when $x = 1$, $x^4 + 2 = 3$. We may now perform the u-substitution:

   $$\int_0^1 x^3 \sin(x^4 + 2) dx = \int_2^3 \sin u \frac{du}{4} = \left. \frac{-\cos u}{4} \right]_2^3 = \frac{\cos 2 - \cos 3}{4}$$

2. **(12 points)** *Answer the following questions about the function $f(x) = 3 \sin \sqrt{x}$.*

   (a) **(4 points)** *What is the domain of $f(x)$?*
   The sine function is defined at all points; however, the square root only exists when its argument is non-negative. Thus, this function ceases to exist when $x < 0$; its domain is thus the points $x \geq 0$, or in interval notation, $[0, \infty)$.

   (b) **(4 points)** *What is the range of $f(x)$?*
   The square-root of x can take on all non-negative values; its sine can thus take on all values between -1 and 1 inclusive. Multiplying by 3, we see that the function $f(x)$ ranges from -3 to 3; we may thus say that the range of $f(x)$ is $-3 \leq f(x) \leq 3$, or, in interval notation, $[-3, 3]$.

   (c) **(4 points)** *Where does the derivative of $f(x)$ exist?*
   Using the chain rule, $f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$. This exists only when its denominator is nonzero and the argument of the square root is non-negative. This eliminates the possibilities $x < 0$ and $x = 0$, leaving only the set of possible x-values $x > 0$, or $(0, \infty)$.

3. **(12 points)** *Let $f(x) = e^{-2x^2}$.*

   (a) **(5 points)** *Where is $f(x)$ increasing? Where is it decreasing?*
   Using the chain rule, $f'(x) = -4xe^{-2x^2}$. Since e^{-2x^2} is positive everywhere, this expression's sign is determined wholly by the sign of $-4x$, which is positive for $x < 0$ and negative for $x > 0$. Thus, $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$.

(b) **(7 points)** *Determine $f(x)$'s concavity and identify points of inflection.*
Using the product rule and chain rule on $f'(x)$, we find that $f''(x) = -4e^{-2x^2} + 16x^2e^{-2x^2} = (16x^2 - 4)e^{-2x^2}$. As above, since e^{-2x^2} is positive everywhere, this expression's sign is determined wholly by the sign of $16x^2 - 4$, which is positive for $x^2 > \frac{1}{4}$ or $|x| > \frac{1}{2}$, and negative for $x^2 < \frac{1}{4}$ or $|x| < \frac{1}{2}$. Thus, $f(x)$ is concave up for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$, and concave down for $-\frac{1}{2} < x < \frac{1}{2}$, and since there are transitions in concavity at these two points, it is evident that $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are points of inflection.

4. **(12 points)** *We are planning to design a rectangular, fenced garden. The front of the garden should be unfenced; the back needs a picket fence, and the sides should be fenced with chain-link. Chain-link fence costs \$8 per foot; a picket fence costs \$10 per foot. We have \$640 to spend on fencing. What dimensions should we choose for our garden to maximize its area?*

Let the garden's width and depth be x and y. Then, we have three fences: two chainlink fences of length y, and a picket fence of length x, so that our cost is $2(8y) + 10x$. Thus, our given situation is that, given $10x + 16y = 640$, what values of x and y maximize xy?

Since $10x + 16y = 640$, $y = 40 - \frac{5}{8}x$ (alternatively, x may be put in terms of y), so we may express the area of the garden as

$$A(x) = x\left(40 - \frac{5}{8}x\right) = 40x - \frac{5}{8}x^2$$

Note that the domain of this function is $0 \leq x \leq 64$; the boundary cases result since, clearly our width must be positive, and our width can be no more than 64 feet since a 64-foot long picket fece will completely exhaust our \$640 budget. We may reject these domain-edge cases out of hand; each of them results in a zero-area garden. Thus, the maximum must be an extremum of $A(x)$ in the domain. We find $A(x)$'s critical points as such:

$$\begin{aligned} A'(x) &= 40 - \frac{5}{4}x \\ 0 &= 40 - \frac{5}{4}x \\ \frac{5}{4}x &= 40 \\ x &= 32 \end{aligned}$$

Since $x = 32$ is our only critical point, our sole extremum (and thus our optimizing dimension) is $x = 32$; our y-dimension is $y = 40 - \frac{5}{8} \cdot 32 = 20$, so our optimal garden plot is $32' \times 20'$.

5. **(12 points)** *A thirteen-foot-long ladder is leaning against a wall. The bottom of the ladder, which is currently five feet away from the wall, is slipping away from the wall at a rate of two feet per hour.*

(a) **(6 points)** *How quickly is the top of the ladder sliding down the wall?*
Let the base of the ladder's distance from the wall in feet be x; let the top of the ladder's height in feet be y; let t be time in hours. Since the base of the ladder is slipping away from the wall, x is increasing at 2 feet per hour; that is, $\frac{dx}{dt} = 2$. Furthermore, the ladder, the ground, and the wall form a right triangle with the ladder as hypotenuse, so by the Pythagorean Theorem

$$x^2 + y^2 = 13^2$$

Since we want $\frac{dy}{dt}$, we differentiate this expression with respect to t to get:

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= \frac{d}{dt}169 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{\frac{dx}{dt}x}{y} \end{aligned}$$

We know that $x = 5$ is given to us, and $y = \sqrt{13^2 - 5^2} = 12$, so $\frac{dy}{dt} = -\frac{2\cdot 5}{12} = -\frac{5}{6}$. Thus the ladder is slipping down the wall (i.e. y is decreasing) at $\frac{5}{6}$ feet per hour.

(b) **(6 points)** *How quickly is the angle between the ladder and the ground changing?*

Here we have the same situation, but must label the angle between the ladder and the ground: we shall call it θ. We know from the labeling of the sides of the triangle that $\cos\theta = \frac{x}{13}$. Thus, differentiating both sides of this equation with respect to t, $(-\sin\theta)\frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{13}$, so

$$\frac{d\theta}{dt} = \frac{-\frac{dx}{dt}}{13\sin\theta} = \frac{-\frac{dx}{dt}}{13\frac{y}{13}} = \frac{-2}{12}$$

so the angle between the ladder in the ground is decreasing (note negative sign on $\frac{d\theta}{dt}$) by $\frac{1}{6}$ radian per hour.

6. **(12 points)** *Consider the function* $g(x) = \frac{(x^2+6x+9}{x-2}$.

(a) **(5 points)** *Identify zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which.*

The rational expression factors into $\frac{(x+3)^2}{x-2}$, so the only zero of this function is at $x = -3$, while the vertical asymptote occurs when the denominator is zero, at $x = 2$. In the long term, this function's dominant terms in the numerator and denominator are x^2 and x respectively, so $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

(b) **(5 points)** *Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.*

Using the quotient rule,

$$\begin{aligned} g'(x) &= \frac{(x-2)\left(\frac{d}{dx}(x^2+6x+9)\right) - (x^2+6x+9)\left(\frac{d}{dx}(x-2)\right)}{(x-2)^2} \\ &= \frac{(x-2)(2x+6) - (x^2+6x+9)(1)}{(x-2)^2} \\ &= \frac{(2x^2+2x-12) - (x^2+6x+9)}{(x-2)^2} \\ &= \frac{x^2-4x-21}{(x-2)^2} \\ &= \frac{(x-7)(x+3)}{(x-2)^2} \end{aligned}$$

While the zero of the denominator might be considered a critical point, it is not a local extremum, since the function $g(x)$ is in fact discontinuous there. Thus, the only extrema we need worry about are the zeroes of $x^2 - 4x - 21$. These occur at $x = 7$ and $x = -3$. Since $g'(x)$ is a quotient of $x^2 - 4x - 21$ and a known positive number, $g'(x)$ has the same sign as $x^2 - 4x - 21$. Thus it is negative between its zeroes and positive off to the sides; -3 is a transition from positive to negative and is thus a maximum; 7 is the opposite and thus a minimum.

(c) **(2 points)** *Which if any of the critical points identified above are global maxima or global minima? Show work or explain.*

None of them are global, because the asymptotes induce arbitrarily large and arbitrarily negative values of $g(x)$; thus, this function has no global maximum or minimum.

7. **(12 points)** *An alien spacecraft is heated to* $1700°F$ *by entry into the atmosphere and crash-lands on an* $100°F$ *day during the summer. After 20 minutes, the ship's hull has cooled to* $1300°F$.

(a) **(4 points)** *Construct a function modeling the temperature of the spacecraft t minutes after impact.*

By Newton's Law of Cooling, the model for the temperature is $T(t) = A + Ce^{kt}$ where A is the ambient temperature, which is in this case 100°F; C and k must be determined from the situation described. We know that $T(0) = 1700$ and that $T(20) = 1300$, so we substitute each of these into the model to determine C and k:

$$\begin{aligned} T(0) &= 100 + Ce^{0k} \\ 1700 &= 100 + C \\ 1600 &= C \end{aligned}$$

And likewise we solve for k:

$$\begin{aligned} T(20) &= 100 + 1600e^{20k} \\ 1300 &= 100 + 1600e^{20k} \\ \frac{3}{4} = \frac{1200}{1600} &= e^{20k} \\ \ln\frac{3}{4} &= 20k \\ \frac{1}{20}\ln\frac{3}{4} &= k \end{aligned}$$

So our model is $T(t) = 100 + 1600e^{\frac{\ln\frac{3}{4}}{20}t}$.

(b) **(4 points)** *The scientific survey team can begin their experiments as soon as the ship has cooled to* $500°F$. *How long will they need to wait after impact to do so?*

We solve for the value of t such that $T(t) = 500$:

$$\begin{aligned} 500 &= 100 + 1600e^{\frac{\ln \frac{3}{4}}{20}t} \\ 400 &= 1600e^{\frac{\ln \frac{3}{4}}{20}t} \\ \frac{1}{4} &= e^{\frac{\ln \frac{3}{4}}{20}t} \\ \ln \frac{1}{4} &= \frac{\ln \frac{3}{4}}{20}t \\ \frac{20 \ln \frac{1}{4}}{\ln \frac{3}{4}} &= t \end{aligned}$$

Thus it cools to 500°F in $\frac{20 \ln \frac{1}{4}}{\ln \frac{3}{4}} \approx 96$ minutes.

(c) **(4 points)** *How quickly is the spacecraft cooling 10 minutes after impact?*

This question is asking merely for the value of $T'(10)$, which might be expected to be negative, as it reflects a decrease in temperature. Since $T'(t) = 100 + 1600e^{\frac{\ln \frac{3}{4}}{20}t}$, it is simple to see that $T'(10) = (80 \ln \frac{3}{4})e^{\frac{\ln \frac{3}{4}}{20}10} = 80 \ln \frac{3}{4} e^{\frac{\ln \frac{3}{4}}{2}} \approx -19$°F per minute.

8. **(12 points)** *The equation $(x-1)(x^2+y^2) = 5x^2$ describes a figure known as the* conchoid of de Sluze*.*

(a) **(9 points)** *Find a formula on the conchoid for $\frac{dy}{dx}$ in terms of x and y.*

We differentiate both sides with respect to x, and use the product and chain rule as appropriate:

$$\begin{aligned} \frac{d}{dx}\left((x-1)(x^2+y^2)\right) &= \frac{d}{dx}5x^2 \\ \left(\frac{d}{dx}(x-1)\right)(x^2+y^2) + (x-1)\left(\frac{d}{dx}(x^2+y^2)\right) &= 10x \\ 1 \cdot (x^2+y^2) + (x-1)(2x+2yy') &= 10x \\ x^2+y^2+(x-1)(2x)+(x-1)2yy' &= 10x \\ (x-1)2yy' &= 10x - x^2 - y^2 - (x-1)(2x) \\ y' &= \frac{12x - 3x^2 - y^2}{2(x-1)y} \end{aligned}$$

(b) **(3 points)** *Find the equation of the tangent line to the conchoid at $(2,4)$.*

Using the result above at this point, we find that at $(2,4)$, the slope of the tangent line is $\frac{dy}{dx} = \frac{12 \cdot 2 - 3 \cdot 2^2 - 4^2}{2(2-1)4} = \frac{-4}{8} = \frac{-1}{2}$.

The equation of the tangent line is thus, in point-slope form, $(y-4) = \frac{-1}{2}(x-2)$.

9. **(12 points)** *Answer the following questions:*

(a) **(4 points)** *Find* $\frac{d}{dx}\cos\left(e^{\sec x}\right)$.

This is a chain rule problem with a chain several layers deep, so we give each part a name. Let $y = \cos u$, $u = e^v$, and $v = \sec x$, so that the expression asked for in the problem is simply $\frac{dy}{dx}$. Using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} \\ &= (-sinu)e^v(\sec x \tan x) \\ &= (-sine^v)e^v(\sec x \tan x) \\ &= (-sine^{\sec x})e^{\sec x}(\sec x \tan x)\end{aligned}$$

(b) **(4 points)** *Given* $g(r) = \frac{d}{dr}\arcsin(r \ln r)$*, find* $g'(r)$.

Let $u = r \ln r$, so that $g'(r) = \frac{d}{dr}\arcsin u$, which with the application of the chain rule becomes $\frac{du}{dr}\frac{d}{du}\arcsin u = \frac{du}{dr}\frac{1}{\sqrt{1-u^2}}$. Then, using the product rule, we find that $\frac{du}{dr} = \ln r + \frac{1}{r}r = \ln r + 1$, so this expression is

$$(\ln r + 1)\frac{1}{\sqrt{1-u^2}} = \frac{\ln r + 1}{\sqrt{1-(r \ln r)^2}}$$

(c) **(4 points)** *Find* $\frac{d}{d\theta}\int 3\ln|\tan\theta| d\theta$.

Let $f(\theta) = 3\ln|\tan\theta|$, so that this expression is simply $\frac{d}{dx}\int f(\theta)d\theta$. Since an indefinite integral is identical to the general antiderivative, this expression is $\frac{d}{dx}\left(F(\theta) + C\right)$, which is simply $f(x)$, so this expression is just $3\ln|\tan\theta|$.

10. **(12 points)** *Determine the following limits.*

(a) **(4 points)** *Using the difference quotient, find the derivative with respect to* x *of* $f(x) = 2x^2 - 3x + 1$*. You may not use L'Hôpital's rule for this problem.*

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)^2 - 3(x+h) + 1 - (2x^2 - 3x + 1)}{h} \\ &= \lim_{h\to 0}\frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 1 - 2x^2 + 3x - 1}{h} \\ &= \lim_{h\to 0}\frac{4xh + 2h^2 - 3h}{h} \\ &= \lim_{h\to 0} 4x + 2h - 3 = 4x - 3\end{aligned}$$

(b) **(4 points)** *Evaluate* $\lim_{\theta\to 0}\frac{\theta\sin\theta}{\theta^3 - \theta^2}$ *or demonstrate that it cannot be evaluated.*

Both the numerator and denominator evaluate to zero, so use of L'Hôpital's rule is justified, yielding $\lim_{\theta\to 0}\frac{\sin\theta + \theta\cos\theta}{3\theta^2 - 2\theta}$, which is still a $\frac{0}{0}$ indeterminate form, so using L'Hôpital's rule again, we get $\lim_{x\to 0}\frac{2\cos\theta - \theta\sin\theta}{6\theta - 2} = \frac{2}{-2} = -1$.

(c) **(4 points)** *Evaluate* $\lim_{x\to\infty} \frac{x^2}{x \ln x}$ *or demonstrate that it cannot be evaluated.*

Both the numerator and denominator grow without bound as x grows large, so this is a $\frac{\infty}{\infty}$ indeterminate form. Using L'Hôpital's rule, we get $\lim_{x\to\infty} \frac{2x}{\ln x+1}$, which is still a $\frac{\infty}{\infty}$ form, so once again we use L'Hôpital's rule to get $\lim_{x\to\infty} \frac{2}{\frac{1}{x}} = \lim_{x\to\infty} 2x$, which grows without bound and thus does not exist.