1. **(7 points)** *Find the following features of the function $f(x) = (2x-3)e^{-x}$, labeling which is which: the intervals over which it is increasing, the intervals over which it is decreasing, the locations and types of local extrema, the intervals over which it is concave up, the intervals over which it is concave down, and the points of inflection.*

   Several of the features (increase/decrease, extrema locations and types) depend on the sign of $f'(x)$. We start by calculating, and as far as possible, factoring, $f'(x)$:

$$f'(x) = (2)e^{-x} + (2x-3)\left(-e^{-x}\right) = (2-2x+3)e^{-x} = (5-2x)e^{-x}$$

   So $f'(x)$ is a product of $(5-2x)$ and e^{-x}. The e^{-x} factor is always positive, so in terms of determining the sign of $f'(x)$, it'll be irrelevant. So $f'(x)$'s sign will be determined solely by $(5-2x)$. We know that $5-2x=0$ when $x=\frac{5}{2}$, so either by explicitly testing points or by applying a knowledge of the shape of linear function, we shall see that $f'(x)$ is positive when $x<\frac{5}{2}$, zero when $x=\frac{5}{2}$, and negative when $x>\frac{5}{2}$.

   $f(x)$ is increasing where $f'(x)$ was found to be positive, which is when $x<\frac{5}{2}$. $f(x)$ is decreasing where $f'(x)$ is negative, which is when $x>\frac{5}{2}$. Finally, $f(x)$ may have extrema where $f'(x)$ is zero or nonexistent; thus $x=\frac{5}{2}$ might be an extremum. Looking to its left and right, we see that $f(x)$ increases *to* $\frac{5}{2}$, and decreases *from* it. This is representative behavior of a local maximum, so $x=\frac{5}{2}$ is a local maximum.

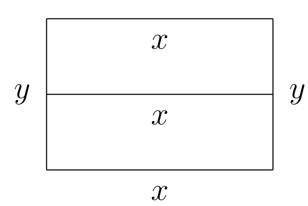
   The remaining features are derived from the sign of $f''(x)$. Using the product rule to determine the derivative of $f'(x) = (5-2x)e^{-x}$:

$$f''(x) = (-2)e^{-x} + (5-2x)\left(-e^{-x}\right) = (-2+2x-5)e^{-x} = (2x-7)e^{-x}$$

   As above, $f''(x)$ is a product of a linear term $(2x-7)$ and e^{-x}. The e^{-x} is again irrelevant from a sign perspective, so $f''(x)$'s sign will be determined solely by $(2x-7)$. We know that $2x-7=0$ when $x=\frac{7}{2}$, so either by explicitly testing points or by applying a knowledge of the shape of linear function, we shall see that $f''(x)$ is negative when $x<\frac{7}{2}$, zero when $x=\frac{7}{2}$, and positive when $x>\frac{5}{2}$.

   Thus, interpreting these in terms of shape characteristics of $f(x)$, it follows that $f(x)$ is concave down when $x<\frac{7}{2}$, concave up when $x>\frac{7}{2}$, and has a point of inflection at $x=\frac{7}{2}$.

2. **(7 points)** *You are placing a fence around all four sides of a farmyard, as well as a fence parallel to the front fence running down the middle. You have 600 feet of fencing. What are the dimensions of the yard of largest area you can design?*



   Above we have drawn this figure, with x denoting the length of the field, and y its height. Adding up all the fences shown, it is clear that this figure uses a length $3x+2y$ of fencing; since we have 600 feet of fence to use in total, it is thus necessary that $3x+2y=600$. This will form a constraint on our optimization problem: the actual optimization problem is to maximize the farmyard's area. The area of a rectangle of dimensions $x \times y$ is xy, so we seek to maximize xy.

Using our constraint, we can put y in terms of x:

$$\begin{aligned} 3x + 2y &= 600 \\ 2y &= 600 - 3x \\ y &= 300 - \frac{3}{2}x \end{aligned}$$

and thus the area which we seek to maximize can be expressed in terms of the single variable x as $A(x) = x(300 - \frac{3}{2}x) = 300x - \frac{3}{2}x^2$. The smallest x could logically be is zero, if the farmyard had no width, and the largest it could be is 200: since there are 600 feet of fencing in total, if all the fencing was used for the 3 horizontal fences, x would be 200.

We thus have a simple optimization problem: find the maximum of $A(x) = 300x - \frac{3}{2}x^2$ over the interval $[0, 200]$. To find critical points, we calculate $A'(x) = 300 - 3x$. This is defined everywhere, but is zero when $x = 100$, so $x = 100$ is a critical point. We thus have three potential maxima: the interval endpoints 0 and 200, and the critical point 100.

It comes as no surprise that $A(0) = A(200) = 0$, since these extrema represent absurd farmyard dimensions (0×300 and 200×0 respectively), and $A(100)$ is positive (specifically, it's 15000), so $x = 100$ is an optimizing choice of x. Then $y = 300 - \frac{3}{2}x = 300 - 150 = 150$, so the optimizing dimensions are 100×150.

3. **(6 points)** *Evaluate the following limits, or demonstrate that they do not exist:*

   (a) **(2 points)** $\lim\limits_{x \to \infty} \frac{x \ln x}{x^2+3}$.

   Note that $\lim_{x\to\infty} x \ln x = +\infty$ and $\lim_{x\to\infty} x^2 - 3 = +\infty$, so the above quotient is a $\frac{\infty}{\infty}$ indeterminate form, and we are justified in invoking L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x \ln x}{x^2+3} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x \ln x)}{\frac{d}{dx}(x^2+3)} = \lim_{x\to\infty} \frac{(1)\ln x + x\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{\ln x + 1}{2x}$$

   Unfortunately, this is still indeterminate: both the numerator and denominator are still increasing without bound as $x \to \infty$, so we may invoke L'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{\ln x + 1}{2x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x + 1)}{\frac{d}{dx}2x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{1}{2x}$$

   This last limit can be easily evaluated to be zero.

   (b) **(2 points)** $\lim\limits_{u \to 0} \frac{e^u - u - 1}{u^2}$.

   Note that $e^0 - 0 - 1 = 0$, and $0^2 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{u\to 0} \frac{e^u - u - 1}{u^2} = \lim_{u\to 0} \frac{\frac{d}{du}(e^u - u - 1)}{\frac{d}{du}u^2} = \lim_{u\to 0} \frac{e^u - 1}{2u}$$

   But this is still a $\frac{0}{0}$ form, since $e^0 - 1 = 0$ and $2 \cdot 0 = 0$, so we use L'Hôpital's rule again:

$$\lim_{u\to 0} \frac{e^u - 1}{2u} = \lim_{u\to 0} \frac{\frac{d}{du}(e^u - 1)}{\frac{d}{du}(2u)} = \lim_{u\to 0} \frac{e^u}{2}$$

   which can be evaluated to be $\frac{e^0}{2} = \frac{1}{2}$.

(c) **(2 points)** $\lim_{\theta \to 0} (\theta^2 - 3\theta) \csc \theta$.

Since $\csc \theta$ has a vertical asymptote at $\theta = 0$, and $0^2 - 3 \cdot 0 = 0$, this limit has a $0 \cdot \infty$ indeterminate form. This form must be reorganized into a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form before L'Hôpital's rule can be used: L'Hôpital only deals with those two specific forms. Fortunately, a quotient-reorganization is implied by the presence of $\csc \theta$, since $\csc \theta = \frac{1}{\sin \theta}$, so we could reconsider this limit as $\lim_{\theta \to 0} \frac{\theta^2 - 3\theta}{\sin \theta}$ which is a straightforward $\frac{0}{0}$ indeterminate form and can be addressed with L'Hôpital's rule:

$$\lim_{\theta \to 0} \frac{\theta^2 - 3\theta}{\sin \theta} = \lim_{\theta \to 0} \frac{\frac{d}{d\theta}(\theta^2 - 3\theta)}{\frac{d}{d\theta} \sin \theta} = \lim_{\theta \to 0} \frac{2\theta - 3}{\cos \theta}$$

which can be directly evaluated to be $\frac{2 \cdot 0 - 3}{\cos 0} = \frac{-3}{1} = -3$.