1. **(10 points)** *A five-digit number must not have a zero as its first digit.*

   (a) **(5 points)** *How many five-digit numbers are there whose digits appear in increasing order (e.g., 25689)?*

   This is a matter of choosing five distinct nonzero digits without regard to order: once the five digits are chosen, the order is necessarily determined, i.e. $\{2,5,6,8,9\}$ can only be ordered as 25689; zero is forbidden since, as the smallest single digit, it would by necessity be first, which would not result in a five-digit number. Thus, the five-digit numbers with digits in increasing order can be placed into a one-to-one correspondence with the 5-element subsets of $\{1,2,3,\ldots,9\}$. There are $\binom{9}{5} = 126$ such sets.

   (b) **(5 points)** *How many five-digit numbers are there whose digits appear in descending order (e.g., 98652)?*

   This situation is exactly as above, except that zero is not forbidden, since as the smallest digit it will appear in the last position. Thus there are $\binom{10}{5} = 252$ such numbers.

2. **(10 points)** *The members of two committees are being drawn from a pool of eight people. Each person can be on either committee or on no committee at all.*

   (a) **(5 points)** *How many ways are there to assign committee-responsibilities to the eight people if there are no restrictions on committee membership?*

   There are 3 ways to assign each person: to committee A, to committee B, or to no committee at all. Since a final assignment of responsibilities is determined by the choices made by 8 distinct individuals, the number of possible assignments is the product of the number of choices possible for each individual: $3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^8$, or 6561 different possible assignations.

   (b) **(5 points)** *How many ways are there to assign committee-responsibilities to the eight people if each committee must have at least one member?*

   From the 3^8 committees above, we shall exclude those in which one or both committees are empty. If committee A is empty, then each individual is assigned to either committee B or neither; since each individual has 2 choices, there are 2^8 total possible assignments. Likewise, if committee B is empty, each individual is assigned to either committee A or neither, and there are 2^8 ways for this to happen. It thus appears that $3^8 - 2^8 - 2^8$ is correct; however, we have in fact excluded the case where both committees are empty *twice*; we need to compensate for our overzealous removal by adding the second removal back in to get $3^8 - 2^8 - 2^8 + 1 = 6050$.

3. **(10 points)** *There are two straightforward ways to prove that* $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

   (a) **(5 points)** *Prove it algebraically, by manipulating the two sides to get equal expressions.*

   $\binom{2n}{2} = \frac{2n(2n-1)}{2} = 2n^2 - n$ and $2\binom{n}{2} + n^2 = 2\frac{n(n-1)}{2} + n^2 = n^2 - n + n^2 = 2n^2 - n$.

(b) **(5 points)** *Prove it combinatorially, by finding a set counted by the left side and demonstrating that the right side counts the same set.*

Consider the set $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. Since $|X| = 2n$, there are $\binom{2n}{2}$ possible 2-element subsets of X. There two-element subsets could be of any of three forms: $\{a_i, a_j\}$ for distinct i and j, $\{b_i, b_j\}$ for distinct i and j, or $\{a_i, b_j\}$ for not necessarily distinct i and j. There are $\binom{n}{2}$ ways to choose two distinct elements of $\{a_1, \ldots, a_n\}$, and likewise for $\{b_1, \ldots, b_n\}$, so the first two cases account for $2\binom{n}{2}$ ways to select two-element sets. In the third case, there are n choices for the index of a_i, and n choices for the index of b_j, so there are n^2 ways to choose such a set. Thus, the number of 2-element subsets of X is $2\binom{n}{2} + n^2$.

4. **(10 points)** *Prove the following identity by demonstrating that the two sides of the equation are counting the same thing:*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

This is actually a variation of the result from the previous problem: consider the set $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. We will enumerate the n-element subsets of X in two different ways. One way is easy: since X has $2n$ elements, there are $\binom{2n}{n}$ ways to choose an n-element subset of it.

Altternatively, we can consider X as consisting of the elements of $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Selecting n elements from X is the same as selecting n elements from the union of A and B. We can represent this by selecting some number of elements from A, and then selecting the remainder from B. If we were to select k elements from A, we could do so in $\binom{n}{k}$ ways; then we would need to select $n - k$ elements from B, which we could do in $\binom{n}{n-k}$ ways. Since a selection of n elements consists of performing each of these selections, there are a total of $\binom{n}{k}\binom{n}{n-k}$ ways to select n elements of X such that exactly k of them are in A. However, any number of elements from 0 to n could be drawn from A, so adding together every seperate case, we have that there are $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ ways to choose n elements of X, so $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.