1. **(10 points)** *Prove the following identity combinatorially:*

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$$

Both sides can easily be interpreted as enumerating the ways to select an element x of $\{1, 2, 3, \ldots, n\}$, and then choose a subset S from among the remaining $n-1$ elements.

The right side of this equation clearly enumerates the set of ways to do the above: there are n choices for x, and, regardless of the choice of x, S is a subset of a set of size $n-1$; thus there are 2^{n-1} choices of S.

To calculate the left side of the equation, we begin by choosing a set T which will contain both S and x, then we select x as an element of T and define $S = T - \{x\}$. Thus, one may uniquely determine x and S by, instead of choosing x then a set S not containing x, choosing a set T and an element thereof to serve as x. For any value of i, there are $\binom{n}{i}$ possible choices of T of size i; then, having schosen a set T of size i, any of its i elements can serve as a value of x, so there are $i\binom{n}{i}$ selections of T and x given $|T| = i$. Ranging over all possible sizes of T, we see that there are $\sum_{i=0}^{n} i\binom{n}{i}$ ways to choose T and x; since there are the same number of choices of (T, x) as there are of (S, x), it follows that $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$.

2. **(10 points)** *If a fair coin is flipped n times, what is the probability that*

   (a) **(3 points)** *The first head comes after exactly m tails?*

   If $m > n-1$, this is impossible (so it has probability zero). Otherwise, the only way it can occur is if the first $m+1$ coinflips are m tails followed by a single tail. The probability is thus $\frac{1}{2^{m+1}}$.

   (b) **(7 points)** *The ith head comes after a total of m previous tails?*

   If $m + i > n$, this is impossible (so it has probability zero). Otherwise, it can occur if the first $m+i-1$ coinflips are m tails and $i-1$ heads, and a head follows this sequence. There are $\binom{m+i-1}{i-1}$ ways for this to occur. The probability is thus $\frac{\binom{m+i-1}{i-1}}{2^{m+i}}$.

3. **(10 points)** *Using paths through a lattice (or some other combinatorial object, if you prefer), prove that the following identity is true for any $m \le k \le n$:*

$$\binom{m+n}{n} = \sum_{i=0}^{m} \binom{k}{i}\binom{m+n-k}{n-i}$$

Consider the lattice paths from $(0, 0)$ to (m, n). If we consider "border stations" at the coordinates $(0, k)$, $(1, k-1)$, $(2, k-2)$, and so forth up to $(m, k-m)$. Note that removal of these border stations from the grid would disconnect $(0, 0)$ from (m, n), so every path from $(0, 0)$ to (m, n) passes through at least one of them; and furthermore, every path passes through no more than one of them, since for $i < j$, $(i, k-j)$ lies above and to the left of $(j, k-j)$, and there is no path which connects a point to a point below it and to its left.

Thus, since every path from $(0,0)$ to (m,n) passes through exactly one border station, the paths from $(0,0)$ to (m,n) can be enumerated as the total of the number of paths through each border station. The number of paths from $(0,0)$ to $(i,k-i)$ to (m,n) is $\binom{k}{i}\binom{m+n-k}{m-i}$, so adding up over all values of i, we see that the total number of lattice paths (which could be otherwise enumerated as $\binom{m+n}{n}$) is $\sum_{i=0}^{m}\binom{k}{i}\binom{m+n-k}{m-i}$.

4. **(10 points)** *Construct generating functions for the number of nonnegative integer solutions to the following equations:*

   (a) **(5 points)** $x_1 + x_2 + x_3 = n$ *for* $x_1 \geq 3$*,* $x_2 \leq 4$*, and* $2 \leq x_3 \leq 5$*.*
   The selection GF for x_1 is $x^3 + x^4 + x^5 + x^6 + \cdots = \frac{x^3}{1-x}$; the selection GF for x_2 is $1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$; the selection GF for x_3 is $x^2 + x^3 + x^4 + x^5 = \frac{x^2-x^6}{1-x}$. The GF representing selection of all three variables is thus:

$$\frac{x^3(1-x^5)(x^2-x^6)}{(1-x)^3}$$

   This can, but need not be, re-expressed in a coefficient-based form:

$$\begin{aligned}\frac{x^3(1-x^5)(x^2-x^6)}{(1-x)^3} &= \frac{x^5 - x^{10} - x^9 + x^{14}}{(1-x)^3} \\ &= \sum_{n=0}^{\infty}\binom{n+2}{2}x^{n+5} - \binom{n+2}{2}x^{n+10} - \binom{n+2}{2}x^{n+9} + \binom{n+2}{2}x^{n+14} \\ &= \sum_{n=0}^{\infty}\left[\binom{n-3}{2} - \binom{n-8}{2} - \binom{n-7}{2} + \binom{n-12}{2}\right]x^n\end{aligned}$$

   (b) **(5 points)** $x_1 + 2x_2 + 5x_3 = n$ *for* $x_3 \leq 2$ *(and no restrictions on the other two).*
   The selection GFs for x_1, x_2, and x_3 are respectively $1 + x + x^2 + \cdots = \frac{1}{1-x}$, $1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1-x^2}$, and $1 + x^5 + x^{10} = \frac{1-x^{15}}{1-x^5}$. The GF enumerating the number of ways to select all three is thus:

$$\frac{1-x^{15}}{(1-x)(1-x^2)(1-x^5)}$$

   It is possible to produce a closed form for the coefficients with the partial fraction decomposition

$$6 + 5x + 4x^2 + 3x^3 + 2x^4 + 2x^5 + x^6 + x^7 + \frac{3}{2(1-x)^2} - \frac{27}{4(1-x)} + \frac{1}{4(1+x)}$$

   but hardly necessary.

5. **(4 point bonus)** *It follows from the binomial theorem that* $\sum_{i=1}^{n}(-1)^n\binom{n}{i} = (1-1)^n = 0$*. Can you find a combinatorial proof of this identity?*