Show work for each problem. Answers without justification, or justified solely by direct enumeration, are not acceptable. Arithmetic expressions may be left unsimplified.

This problem set is due at the beginning of class on *September 17*.

1. **(10 points)** Prove the following identity combinatorially:

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$$

2. **(10 points)** If a fair coin is flipped n times, what is the probability that
   (a) **(3 points)** The first head comes after exactly m tails?
   (b) **(7 points)** The ith head comes after a total of m previous tails?

3. **(10 points)** Using paths through a lattice (or some other combinatorial object, if you prefer), prove that the following identity is true for any $m \leq k \leq n$:

$$\binom{m+n}{n} = \sum_{i=0}^{m} \binom{k}{i}\binom{m+n-k}{m-i}$$

4. **(10 points)** Construct generating functions for the number of nonnegative integer solutions to the following equations:
   (a) **(5 points)** $x_1 + x_2 + x_3 = n$ for $x_1 \geq 3$, $x_2 \leq 4$, and $2 \leq x_3 \leq 5$.
   (b) **(5 points)** $x_1 + 2x_2 + 5x_3 = n$ for $x_3 \leq 2$ (and no restrictions on the other two).

5. **(4 point bonus)** It follows from the binomial theorem that $\sum_{i=1}^{n}(-1)^n\binom{n}{i} = (1-1)^n = 0$. Can you find a combinatorial proof of this identity?

From two letters or forms are composed two dwellings; from three, six; from four, twenty-four; from five, one hundred and twenty; from six, seven hundred and twenty; from seven, five thousand and forty; and thence their numbers increase in a manner beyond counting and are incomprehensible. —*Sefer Yetzira*