1. **(10 points)** *Prove the following combinatorial identities:*

   (a) **(5 points)** *Recall that $S(n,k)$ is equal to the number of ways to subdivide an n-element set into k nonempty parts. Produce a combinatorial argument to show that $S(n,k) = kS(n-1,k) + S(n-1,k-1)$.*

   Let us consider the ways to divide $\{1, 2, \ldots, n\}$ into k nonempty sets. There are two possibilities to be addressed: either n is in a set by itself or with other numbers.

   If n is in a set by itself, then removal of this set $\{n\}$ leaves us with a partition of the numbers $\{1, 2, \ldots, n-1\}$ into $k-1$ sets. There are $S(n-1, k-1)$ such partitions, and each of them can be identified uniquely with a k-partition of $\{1, 2, \ldots, n-1\}$ by adding in the extra set $\{n\}$.

   On the other hand, if n is in a set with other numbers, removal of n from this set leaves k sets behind partitioning the numbers $\{1, 2, \ldots, n-1\}$. There are $S(n-1, k)$ such possible partitions, but they are *not* uniquely identified with k-partitions of $\{1, 2, \ldots, n-1\}$, since the term n could have been removed from (and can be reinserted into) any of the k sets. Thus, there are k different ways to build a k-partition of $\{1, 2, \ldots, n\}$ from a k-partition of $\{1, 2, \ldots, n-1\}$, so this possibility accounts for $k \cdot S(n-1, k)$ partitions.

   Assembling these two cases, we see that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

   (b) **(5 points)** *Prove that for any $m < n$, $\sum_{k=0}^{m} \binom{n}{k,m-k,n-m} = 2^m \binom{n}{m}$.*

   This is actually very much like problem #1 from the previous problem set; it's actually a generalization, since that problem is equivalent to the $m = n-1$ case of this one.

   The left side is fairly easy to interpret: it represents the number of ways to divide an n-element set into three distinct parts of size k, $m-k$, and $n-m$, for any value of k. So, more properly, it represents the number of ways to divide n elements into classes A, B, and C so that classes A and B together contain m elements, and class C has $n-m$.

   The right side can be interpreted as enumerating the same thing, in a different way. We start by selecting the membership of classes A and B together; we want to have m members, so there are $\binom{n}{m}$ ways to select these, and the remaining $n-m$ elements are consigned to class C. Now, we want to freely distribute these m elements between A and B: for each element there are two possibilities, that it is in class A or class B, so there are 2^m ways to distribute the preselected m elements between A and B. Thus, this enumeration is identically $\binom{n}{m}2^m$, which is the right side of the equation.

   This question could also be answered algebraically. It could be shown to be true for all m by virtue of a polynomial representation of each side, i.e. $\sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{k,m-k,n-m} x^m$ and $\sum_{m=0}^{n} 2^m \binom{n}{m} x^m$. Note that we can interestingly restructure the sum in the first form here and use the multinomial theorem:

$$\begin{aligned}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{k,m-k,n-m}x^m &= \sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{k,m-k,n-m}x^m\\ &= \sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k,i,n-i-k}x^{i+k}\\ &= \sum_{i+j+k=n}\binom{n}{k,i,j}x^i x^k 1^j\\ &= (x+x+1)^n\\ &= (2x+1)^n\\ &= \sum_{m=0}^{n}\binom{n}{m}(2x)^m = \sum_{m=0}^{n}2^m\binom{n}{m}x^m\end{aligned}$$

2. **(10 points)** *We know that $p_k(n)$ is the number of partitions of the number n into exactly k nonzero parts, and it has a generating function given by $\sum_{n=0}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)}$.*

   (a) **(5 points)** *Prove that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ by using a direct combinatorial method (e.g. bijection or alternative enumerations of the same set).*

   Given a partition of the integer n into k parts, we have two possibilities: either at least one of the parts is 1, or all of the parts are larger than 1.

   The partitions containing a 1 can be bijectively mapped to the same partitions with the 1 removed. By way of illustration, $p_3(8) = 5$, describing the partitions $6+1+1$, $5+2+1$, $4+3+1$, $4+2+2$, and $3+3+2$. Of these, those with 1s in them are in a bijective map with 2-part partitions of 7: $6+1+1 \leftrightarrow 6+1$, $5+2+1 \leftrightarrow 5+2$, and $4+2+1 \leftrightarrow 4+3$. Thus, the partitions containing a 1 are equal in number to the partitions of $n-1$ into $k-1$ parts; that is, $p_{k-1}(n-1)$.

   On the other hand, partitions without a 1 can be subjected to a different sort of reversible reduction: decrement the value of each part by 1, for a total decrement of k. So as above, we would convert the 8-partitions $4+2+2$ and $3+3+2$ into the 5-partitions $3+1+1$ and $2+2+1$ respectively. Thus, we may bijectively map the partitions which do not contain a 1 to the k-part partitions of $n-k$, of which there are $p_k(n-k)$.

   Adding up these two cases, we see that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.

   (b) **(5 points)** *Prove that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ by equating the generating function $\sum_{n=0}^{\infty} p_k(n)x^n$ to the sum $\sum_{n=0}^{\infty} p_{k-1}(n-1)x^n + \sum_{n=0}^{\infty} p_k(n-k)x^n$.*

   It is know that $\sum_{n=0}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$. We shall show that $\sum_{n=0}^{\infty} p_{k-1}(n-$

$1)x^n + \sum_{n=0}^{\infty} p_k(n-k)x^n$ can be algebraically manipulated to be the same thing.

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{k-1}(n-1)x^n + p_k(n-k)x^n &= x\sum_{n=0}^{\infty} p_{k-1}(n-1)x^{n-1} + x^k \sum_{n=0}^{\infty} p_k(n-k)x^{n-k} \\
&= x\sum_{n=-1}^{\infty} p_{k-1}(n)x^n + x^k \sum_{n=-k}^{\infty} p_k(n)x^n \\
&= x\sum_{n=0}^{\infty} p_{k-1}(n)x^n + x^k \sum_{n=0}^{\infty} p_k(n)x^n \text{ since } p_k(n) = 0 \text{ for } n < 0 \\
&= x\frac{x^{k-1}}{(1-x)\cdots(1-x^{k-1})} + x^k \frac{x^k}{(1-x)\cdots(1-x^k)} \\
&= \frac{x^k(1-x^k)}{(1-x)\cdots(1-x^k)} + \frac{x^{2k}}{(1-x)\cdots(1-x^k)} \\
&= \frac{x^k}{(1-x)\cdots(1-x^k)}
\end{aligned}$$

3. **(15 points)** *We shall determine the number a_n of strings of the numbers 0, 1, or 2 of length n which do not contain two consecutive zeroes.*

   (a) **(5 points)** *Find a recurrence relation for a_n, including initial cases.*

   $a_0 = 1$ to represent the empty string; $a_1 = 3$, since any of the strings 0, 1, or 2 will suffice. For our recurrence, let us consider the ways we can get a 00-free string of length n from a shorter 00-free string. We could, of course, append a 1 or 2 to a string of length $n-1$. We could also append a 10 or 20 to a string of length $n-2$. Thus, $a_n = 2a_{n-1} + 2a_{n-2}$.

   (b) **(5 points)** *Find a closed form for a_n.*

   The characteristic polynomial is $x^2 - 2x - 2$, which has roots $1 \pm \sqrt{3}$, so the general solution to the recurrence alone is $a_n = k(1+\sqrt{3})^n + \ell(1-\sqrt{3})^n$. The initial conditions tell us that $1 = k + \ell$ and $3 = (1+\sqrt{3})k + (1-\sqrt{3})\ell$. Solving this system of equations gives $k = \frac{3+2\sqrt{3}}{6}$ and $\ell = \frac{3-2\sqrt{3}}{6}$, for a solution of

   $$a_n = \frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n$$

   (c) **(5 points)** Find a closed form for the generating function of a_n (you may do this before part (b), if desired, and use it to solve part (b)).

4. **(5 points)** Here F_n denotes the nth Fibonacci number with initial values $1, 1$, so that $F_0 = 1$, $F_1 = 1$, and $F_2 = 2$. Prove that for non-negative m and n, $F_{m+n} = F_mF_n + F_{m-1}F_{n-1}$.

   There are many ways to prove this, but one of the easiest is with the checkers-and-dominoes combinatorial expression of Fibonacci numbers. Let us consider tilings of a $1 \times (m+n)$ checkerboard with checkers and dominoes. Clearly, there are F_{m+n} of these.

Let us consider the result of "cutting" this board into a left half which has dimensions $1 \times m$ and a right half which has dimensions $1 \times n$.

Considering those tilings where the cut divides the board into two seperate tilings; that is, where no tile overlaps the cut, there is clearly a bijection between these tilings and individual seperate tilings of a $1 \times m$ board and a $1 \times n$ board; this is a biejction since a "cut" of a tiling can be reversed by reassembling the pieces. Thus, these tilings are in bijective correspondence with a set of size $F_m F_n$ (the set of pairs of $1 \times m$ and $1 \times n$ tilings).

Now, let us consider those tilings where the cut *does* break a tile. The only tile which spans multiple squares (and can thus cross a cut) is a domino: thus these tilings have a domino between the mth and $(m+1)$th squares. Removing this domino and the squares under it, we see that the original $1 \times (m+n)$ checkerboard is decomposed into a freely-tiled $1 \times (m-1)$ checkerboard, a two-element gap, and a freely tiled $1 \times (n-1)$ checkerboard. Thus these tilings can be placed into a one-to-one correspondence with pairs of tilings of a $1 \times (m-1)$ checkerboard and a $1 \times (n-1)$ checkerboard. Thus, there are $F_{m-1}F_{n-1}$ of these.

Adding together the two cases gives the identity requested.

5. **(5 point bonus)** Let $f(x)$ be a monic polynomial of degree n with integer coefficients and distinct (not necessarily real) roots $r_1, r_2, \ldots, r_n$. Show that $r_1^k + r_2^k + r_3^k + \cdots + r_n^k$ is an integer for any positive integer k.

On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question. —*Charles Babbage*