This problem set is due at the beginning of class on *October 1*.

1. **(10 points)** Prove the following combinatorial identities:
   (a) **(5 points)** Recall that $S(n, k)$ is equal to the number of ways to subdivide an n-element set into k nonempty parts. Produce a combinatorial argument to show that $S(n, k) = kS(n-1, k) + S(n-1, k-1)$.
   (b) **(5 points)** Prove that for any $m < n$, $\sum_{k=0}^{m} \binom{n}{k, m-k, n-m} = 2^m \binom{n}{m}$.

2. **(10 points)** We know that $p_k(n)$ is the number of partitions of the number n into exactly k nonzero parts, and it has a generating function given by $\sum_{n=0}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)}$.
   (a) **(5 points)** Prove that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ by using a direct combinatorial method (e.g. bijection or alternative enumerations of the same set).
   (b) **(5 points)** Prove that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ by equating the generating function $\sum_{n=0}^{\infty} p_k(n)x^n$ to the sum $\sum_{n=0}^{\infty} p_{k-1}(n-1)x^n + \sum_{n=0}^{\infty} p_k(n-k)x^n$.

3. **(15 points)** We shall determine the number a_n of strings of the numbers 0, 1, or 2 of length n which do not contain two consecutive zeroes.
   (a) **(5 points)** Find a recurrence relation for a_n, including initial cases.
   (b) **(5 points)** Find a closed form for a_n.
   (c) **(5 points)** Find a closed form for the generating function of a_n (you may do this before part (b), if desired, and use it to solve part (b)).

4. **(5 points)** Here F_n denotes the nth Fibonacci number with initial values $1, 1$, so that $F_0 = 1$, $F_1 = 1$, and $F_2 = 2$. Prove that for non-negative m and n, $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$.

5. **(5 point bonus)** Let $f(x)$ be a monic polynomial of degree n with integer coefficients and distinct (not necessarily real) roots $r_1, r_2, \ldots, r_n$. Show that $r_1^k + r_2^k + r_3^k + \cdots + r_n^k$ is an integer for any positive integer k.

> *On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.*
> —*Charles Babbage*