1. **(10 points)** *For this problem, it will be helpful to note the following two power series expansions:*

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots \qquad \frac{e^x-e^{-x}}{2} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$$

(a) **(5 points)** *Find an exponential generating function for the sequence $\{a_n\}$ of the number of ways to build an n-letter string consisting of As, Bs, Cs, and Ds such that there is at least one A, an even number of Bs, an odd number of Cs, and any number of Ds.*

The exponential generating function for selecting one or more As is $x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots = e^x-1$. The EGF for selecting an even number of Bs is $1+\frac{x^2}{2}+\frac{x^4}{24}+\cdots = \frac{e^x+e^{-x}}{2}$. The EGF for selecting an odd number of Cs is $x+\frac{x^3}{6}+\frac{x^5}{120}+\cdots = \frac{e^x-e^{-x}}{2}$. The EGF for selecting an arbitrary number of Ds is $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots = e^x$. Assemblnig all these into the EGF for the procedure as a whole:

$$(e^x-1)\frac{e^x+e^{-x}}{2}\frac{e^x-e^{-x}}{2}e^x = \frac{e^{4x}-e^{3x}-1+e^{-3x}}{4}$$

The expansion of the product is not strictly necessary here, but will be useful for the next part of this problem.

(b) **(5 points)** *Using your exponential generating function, find a formula for a_n.*

$$\begin{aligned}\frac{e^{4x}-e^{3x}-1+e^{-3x}}{4} &= \frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{(4x)^n}{n!}-\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}-1+\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}\right]\\ &= -\frac{1}{4}+\sum_{n=0}^{\infty}\frac{4^n-3^n+(-1)^n}{4}\frac{x^n}{n!}\end{aligned}$$

So a_0 will be somewhat a special case, due to the hanging constant term in the EGF; $a_0 = \frac{4^0-3^0+(-1)^0}{4}-\frac{1}{4}=0$, for every other value of n, $a_n = \frac{4^n-3^n+(-1)^n}{4}$.

2. **(20 points)** *A string is called "excellent" if it consists of any number of As, any number of Bs, and exactly one C. Let a_n represent the number of excellent strings of length n.*

(a) **(5 points)** *Construct an exponential generating function $g(x)=\sum_{n=0}^{\infty} a_n\frac{x^n}{n!}$.*

The EGFs for selecting As, Bs, and Cs are respectively e^x, e^x, and x. Thus $g(x)=e^xe^xx = xe^{2x}$.

(b) **(5 points)** *Using casewise analysis on the first term of an excellent string, find a recurrence relation, with initial conditions, for a_n.*

An excellent string of length n may begin with an A or B, in which case the $n-1$ letters that follow must contain exactly one C; that is, an excellent string of length n can be produced by suffixing an A or B with an excellent string of length $n-1$. There are 2 ways to choose the prefix and a_{n-1} possible excellent strings which follow it, for a total of $2a_{n-1}$ excellent strings of length n generated this way.

On the other hand, if we start with a C, then any string of $n-1$ A and Bs following this are an excellent string, so there are 2^{n-1} excellent strings of length n beginning with C.

Assembling these two cases, we see that $a_n = 2a_{n-1} + 2^{n-1}$. Our initial case is $a_0 = 0$, since an empty string, which does not contain a C, cnnot be excellent.

(c) **(5 points)** *From your recurrence relation, determine the closed form of the ordinary generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.*

Starting from the variant form $a_{n+1} = 2a_n + 2^n$ for $n \geq 0$, we will find that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+1}x^n &= \sum_{n=0}^{\infty}(2a_n x^n + 2^n)x^n \\
\sum_{n=0}^{\infty} a_{n+1}x^{n+1} &= 2x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty}(2x)^n \\
\sum_{n=1}^{\infty} a_n x^n &= 2xf(x) + x\frac{1}{1-2x} \\
f(x) - a_0 &= 2xf(x) + x\frac{1}{1-2x} \\
(1-2x)f(x) &= x\frac{1}{1-2x} \\
f(x) &= \frac{x}{(1-2x)^2}
\end{aligned}$$

(d) **(5 points)** *Determine a formula for a_n, using any method you wish (direct enumeration, OGF or EGF coefficient-determination, or recurrence solution).*

This is an open-ended question with several legitimate approaches. The four most likely are shown below.

Direct enumeration: An excellent string of length n consists of exactly one C and any number of As and Bs. There are n locations for the C; after one is chosen, the remaining $n-1$ locations can be filled with As and Bs freely, so since we have two possible choices for each location, there are 2^{n-1} ways to fill the other lcoations. Thus, there are $n2^{n-1}$ excellent strings of length n.

Ordinary generating function coefficient-determination: As seen in part (c), $\sum_{n=0}^{\infty} a_n x^n = \frac{x}{(1-2x)^2}$. Using partial fractions, we can find this to be $\frac{1}{2(1-2x)^2} -$

$\frac{1}{2(1-2x)}$. Then:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= \frac{1}{2(1-2x)^2} - \frac{1}{2(1-2x)} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+2-1}{2-1}(2x)^n - \frac{1}{2}\sum_{n=0}^{\infty}(2x)^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(2x)^n - \frac{1}{2}\sum_{n=0}^{\infty}(2x)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{2}n2^n x^n\end{aligned}$$

so $a_n = \frac{1}{2}n2^n$.

Exponentian generating function coefficient-determination: As seen in part (a), $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = xe^{2x}$. Using the known power-series representation of an exponential:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= xe^{2x} \\ &= x\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} \\ &= \sum_{n=0}^{\infty} 2^n \frac{x^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} 2^n (n+1)\frac{x^{n+1}}{(n+1)!} \\ &= \sum_{n=1}^{\infty} 2^{n-1} n \frac{x^n}{n!}\end{aligned}$$

so $a_n = 2^{n-1}n$ for $n > 0$, and $a_0 = 0$.

Recurrence solution: The recurrence relation determined in part (b) is a non-homogeneous recurrence relation with nonhomogeneous term of $\frac{1}{2}2^n$. We start by solving the associated homogeneous recurrence $b_n = 2b_{n-1}$, which has characteristic polynomial $x - 2$ and thus has general solution $b_n = k2^n$.

Since the nonhomogeneous term in the recurrence for a_n is 2^n, our immediate choice for a particular solution to this recurrence is $a_n^P = A2^n$; however, since this overlaps the terms in the homogeneous solution, we must multiply by n to get the prospective particular solution $a_n^P = An2^n$. Plugging this into the original recurrence, we find:

$$\begin{aligned} a_n^P &= 2a_{n-1}^P + 2^{n-1} \\ An2^n &= 2A(n-1)2^{n-1} + 2^{n-1} \\ 2An &= 2A(n-1) + 1 \\ 2A &= 1 \longrightarrow A = \frac{1}{2} \end{aligned}$$

so $a_n^P = \frac{1}{2}n2^n$ satisfies the recurrence relation; to get the general solution to the recurrence, we add the general solution to the associated homogeneous differential equation, yielding $a_n = a_n^P + b_n = \frac{1}{2}n2^n + k2^n$. We now determine k by demanding that the initial condition be satisfied: $0 = a_0 = \frac{1}{2} \cdot 02^0 + k2^0$, which gives $k = 0$. Thus $a_n = \frac{1}{2}n2^n + 0 \cdot 2^n = \frac{1}{2}n2^n$.

3. **(10 points)** Consider strings with elements from $\{0, 1, 2\}$. A string is "happy" if it contains neither the sequence "00" nor "11". Let a_n represent the number of happy strings of length n ending in 0; let b_n represent the number of happy strings of length n ending in 1; let c_n represent the number of happy strings of length n ending in 2. Note that, using the bijection of exchanging the roles of zeroes and ones, it is easy to see that $a_n = b_n$.

   (a) **(5 points)** Produce a system of simultaneous recurrence relations describing a_n, b_n and c_n, and their initial conditions. Simplify your answer to a system in only a_n and c_n by making use of the fact that $a_n = b_n$.

   We can get a happy string ending in 0 by adding a 0 to the end of a happy string ending in 1 or 2, so $a_n = b_{n-1} + c_{n-1}$. Likewise, a happy string ending in 1 results from adding a 1 to the end of a happy string ending in 0 or 2, to $b_n = a_{n-1} + c_{n-1}$. A happy string ending in 2 can result from extending *any* happy string, however, so $c_n = a_{n-1} + b_{n-1} + c_{n-1}$.

   Using $a_n = b_n$, we can reduce this to a system of two equations:

$$\begin{cases} a_n = a_{n-1} + c_{n-1} \\ c_n = 2a_{n-1} + c_{n-1} \end{cases}$$

   Initial conditions are somewhat problematic. It is easy to argue that $a_1 = 1$ and $c_1 = 1$, but a_0 and c_0 are definitionally peculiar, since a string of length zero cannot be said to end in any digit. The choice of $a_0 = 0$ and $c_0 = 1$ gives correct values for a_1 and c_1, but is not particularly defensible on enumerative grounds (it could be more strongly defended if a_n were defined as those strings which end in a "0" — which the null string does not — and c_n were defined as those strings not ending in a "0" or "1", which the null string indeed satisfies).

   For calculation purposes below, we shall make use of $a_0 = 0$ and $c_0 = 1$.

   (b) **(5 points)** Find formulas for a_n and c_n.

   There are two straightforward methods for solving this, and there may be other approaches as well.

Ordinary generating functions: If $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}c_nx^n$, then we may find relationships between $f(x)$ and $g(x)$ as such:

$$\begin{aligned}\sum_{n=0}^{\infty}a_{n+1}x^n &= \sum_{n=0}^{\infty}(a_n+c_n)x^n\\ \sum_{n=0}^{\infty}a_{n+1}x^{n+1} &= x\sum_{n=0}^{\infty}a_nx^n + x\sum_{n=0}^{\infty}c_nx^n\\ f(x)-a_0 &= xf(x)+xg(x)\\ (1-x)f(x) &= xg(x)\\ f(x) &= \frac{x}{1-x}g(x)\end{aligned}$$

and similarly:

$$\begin{aligned}\sum_{n=0}^{\infty}c_{n+1}x^n &= \sum_{n=0}^{\infty}(2a_n+c_n)x^n\\ \sum_{n=0}^{\infty}c_{n+1}x^{n+1} &= 2x\sum_{n=0}^{\infty}a_nx^n + x\sum_{n=0}^{\infty}c_nx^n\\ g(x)-c_0 &= 2xf(x)+xg(x)\\ (1-x)g(x) &= 2xf(x)+1\\ g(x) &= \frac{2xf(x)+1}{1-x}\end{aligned}$$

Putting these together, we see that

$$f(x)=\frac{x\frac{2xf(x)+1}{1-x}}{1-x}=\frac{2x^2}{(1-x)^2}f(x)+\frac{x}{(1-x)^2}$$

which, solved for $f(x)$, gives $f(x)=\frac{x}{1-2x-x^2}$; the partial fraction decomposition of this is $\frac{\frac{\sqrt{2}}{4}}{1-(1+\sqrt{2})x}-\frac{\frac{\sqrt{2}}{4}}{1-(1-\sqrt{2})x}$. We can rephrase these as infinite series to wrangle their coefficients out:

$$\begin{aligned}\sum_{n=0}^{\infty}a_nx^n &= \frac{\frac{\sqrt{2}}{4}}{1-(1+\sqrt{2})x}-\frac{\frac{\sqrt{2}}{4}}{1-(1-\sqrt{2})x}\\ &= \frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\left[(1+\sqrt{2})x\right]^n-\frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\left[(1-\sqrt{2})x\right]^n\\ &= \sum_{n=0}^{\infty}\frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]x^n\end{aligned}$$

so $a_n=\frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$. Similarly, $g(x)=\frac{1-x}{x}f(x)=\frac{1-x}{1-2x-x^2}$, with partial fraction decomposition $\frac{\frac{1}{2}}{1-(1+\sqrt{2})x}+\frac{\frac{1}{2}}{1-(1-\sqrt{2})x}$, which can be used to find that $c_n=\frac{1}{2}\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n\right]$

Exponential generating functions: If $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$, then noting that $f'(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$ and $g'(x) = \sum_{n=0}^{\infty} c_{n+1} \frac{x^n}{n!}$, the above simultaneous linear recurrence relations can be converted into simultaneous linear differential equations:

$$\begin{cases} f'(x) = f(x) + g(x) \\ g'(x) = 2f(x) + g(x) \end{cases}$$

which, if we establish $\mathbf{v} = \left[\begin{smallmatrix} f(x) \\ g(x) \end{smallmatrix}\right]$, becomes the matrix differential equation $\mathbf{v}' = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right] \mathbf{v}$. The eigenvalues of this differential equation are $1+\sqrt{2}$ and $1-\sqrt{2}$, with respective eigenvectors $(1, \sqrt{2})$ and $(1, -\sqrt{2})$. Thus, the general solution to this matrix differential equation is:

$$\mathbf{v} = ke^{(1+\sqrt{2})x} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + \ell e^{(1-\sqrt{2})x} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

so $f(x) = ke^{(1+\sqrt{2})x} + \ell e^{(1-\sqrt{2})x}$ while $g(x) = \sqrt{2}ke^{(1+\sqrt{2})x} - \sqrt{2}\ell e^{(1-\sqrt{2})x}$. From the initial conditions, we know that $f(0) = 0$ and $g(0) = 1$, yielding the system of equations:

$$\begin{cases} k + \phantom{\sqrt{2}}\ell = 0 \\ \sqrt{2}k - \sqrt{2}\ell = 1 \end{cases}$$

which gives $k = \frac{\sqrt{2}}{4}$ and $\ell = \frac{-\sqrt{2}}{4}$. Thus:

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{4} e^{(1+\sqrt{2})x} - \frac{\sqrt{2}}{4} e^{(1+\sqrt{2})x} \\ \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{[(1+\sqrt{2})x]^n}{n!} - \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{[(1-\sqrt{2})x]^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{\sqrt{2}}{4}(1+\sqrt{2})^n - \frac{\sqrt{2}}{4}(1+\sqrt{2})^n \right] \frac{x^n}{n!} \end{aligned}$$

giving a formula for a_n. Similarly:

$$\begin{aligned} g(x) &= \frac{1}{2} e^{(1+\sqrt{2})x} + \frac{1}{2} e^{(1+\sqrt{2})x} \\ \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{[(1+\sqrt{2})x]^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{[(1-\sqrt{2})x]^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1+\sqrt{2})^n \right] \frac{x^n}{n!} \end{aligned}$$

giving a formula for c_n.