This problem set is due at the beginning of class on *October 22*.

1. **(10 points)** For this problem, it will be helpful to note the following two power series expansions:

$$\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad \frac{e^x-e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

   (a) **(5 points)** Find an exponential generating function for the sequence $\{a_n\}$ of the number of ways to build an n-letter string consisting of As, Bs, Cs, and Ds such that there is at least one A, an even number of Bs, an odd number of Cs, and any number of Ds.

   (b) **(5 points)** Using your exponential generating function, find a formula for a_n.

2. **(20 points)** A string is called "excellent" if it consists of any number of As, any number of Bs, and exactly one C. Let a_n represent the number of excellent strings of length n.

   (a) **(5 points)** Construct an exponential generating function $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

   (b) **(5 points)** Using casewise analysis on the first term of an excellent string, find a recurrence relation, with initial conditions, for a_n.

   (c) **(5 points)** From your recurrence relation, determine the closed form of the ordinary generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

   (d) **(5 points)** Determine a formula for a_n, using any method you wish (direct enumeration, OGF or EGF coefficient-determination, or recurrence solution).

3. **(10 points)** Consider strings with elements from $\{0, 1, 2\}$. A string is "happy" if it contains neither the sequence "00" nor "11". Let a_n represent the number of happy strings of length n ending in 0; let b_n represent the number of happy strings of length n ending in 1; let c_n represent the number of happy strings of length n ending in 2. Note that, using the bijection of exchanging the roles of zeroes and ones, it is easy to see that $a_n = b_n$.

   (a) **(5 points)** Produce a system of simultaneous recurrence relations describing a_n, b_n and c_n, and their initial conditions. Simplify your answer to a system in only a_n and c_n by making use of the fact that $a_n = b_n$.

   (b) **(5 points)** Find formulas for a_n and c_n.

Human intelligence is a product of analogy and combinatorics. Analogy allows the mind to use a few innate ideas—space, force, essence, goal—to understand more abstract domains. Combinatorics allows an a finite set of simple ideas to give rise to an infinite set of complex ones.
—Steven Pinker