This problem set is due at the beginning of class on *November 10*.

1. **(5 points)** *Find an asymptotically accurate approximation for* $\binom{n^2}{n}$ *in terms of polynomials, exponentials, and self-exponentials. You may write it in big-O notation if you wish.*

   By Stirling's approximation,

$$\binom{n^2}{n} \approx \frac{1}{\sqrt{2\pi}} \frac{(n^2)^{n^2+1/2}}{(n^2-n)^{n^2-n+1/2} n^{n+1/2}}$$

   We can simplify this a great geal with some simple algebra:

$$\begin{aligned}\binom{n^2}{n} &\approx \frac{1}{\sqrt{2\pi}} \frac{n^{2n^2+1}}{(n(n-1))^{n^2-n+1/2} n^{n+1/2}} \\ &\approx \frac{1}{\sqrt{2\pi}} \frac{n^{2n^2+1}}{(n-1)^{n^2-n+1/2} n^{n^2+1}} \\ &\approx \frac{1}{\sqrt{2\pi}} \frac{n^{(n^2)}}{(n-1)^{n^2-n+1/2}} \\ &\approx \frac{(n-1)^{n-1/2}}{\sqrt{2\pi}} \left(\frac{n}{n-1}\right)^{(n^2)}\end{aligned}$$

   The first term in this product is clearly $\Theta(\frac{n^n}{\sqrt{n}})$. The second term is less obvious — asymptotically it is a $1^\infty$ indeterminate form, and experimental verification suggests it grows very quickly. On the other hand, we know that $\binom{n^2}{n} < \frac{(n^2)^n}{n!} = O(n^{2n})$. Thus, we can, with Stirling's approximation, get the following (weak) asymptotic bounds:

$$\Omega(\frac{n^n}{\sqrt{n}}) = \binom{n^2}{n} = O(n^{2n})$$

2. **(10 points)** *You have a large supply of beads of 4 different colors and want to string eight of them on a necklace, making use of each bead at least once. How many ways are there to do so, if necklaces are considered identical if they are rotations or reflections of each other?*

   The set $S$ of necklaces prior to symmetry-identification is the set of ordered choices of 8 items from a set of size 4, such that each item is chosen at least once. Then $|S|$ is the enumeration statistic $4!S(8,4) = \binom{4}{0}4^8 - \binom{4}{1}3^8 + \binom{4}{2}2^8 - \binom{4}{3}1^8 + \binom{4}{4}0^8 = 40824$.

   However, we seek $|S/D_8|$, where $D_8$ is the dihedral group consisting of the identity, 7 rotations, and 8 reflections. We will invoke Burnside's lemma to find the invariants of each of these permutations.

   Every bead-coloring would remain fixed under the identity, by definition, so its invariant set is exactly $S$ itself, so it has $4!S(8,4)$ invariants.

Under any of the odd rotations, $r$, $r^3$, $r^5$, or $r^7$, one may trace the positions of each bead to note that the beads are mapped to eachother in an 8-bead cycle, in which each bead would have to have the same color as its successor, so every bead would have to be the same color; there are no elements of $S$ which match this condition, since every coloring in $S$ uses all 4 colors. Thus $r$, $r^3$, $r^5$, and $r^7$ have no invariants (formally, one might say they have $4!S(1,4)$ invariants, which happens to be zero).

Under the rotations $r^2$ and $r^6$, the 8 beads are decomposed into two 4-bead cycles, so any coloring that used the same color for all the odd beads and the same color for all the even beads would be invariant under $r^2$ or $r^6$. However, such a coloring would only use two colors, and thus not be in $S$ in the first place; these rotations also have no invariants (formally, one might say they have $4!S(2,4)$ invariants, which happens to be zero).

The rotation $r^4$ swaps pairs of opposite beads, so an invariant of $r_4$ consists of four beads followed by four beads in the same color pattern as the first four. We thus produce an invariant of $r^4$ by choosing four beads in order, using all four colors: we can do so in $4!S(4,4) = 24$ ways.

Similarly, any of the four flips which move every bead will also have an invariant defined by selecting the four beads on one side of the axis of reflection, and then forcing the beads on the other side to be colored based on the colors already chosen, so these flips also have $4!S(4,4) = 24$ invariants each.

Lastly, there are the four flips where the axes of reflection pass through opposite beads. These flips leave two beads unmoved, qand swap the remaining 3 pairs. Thus we have 5 colorable structures on an invariant: one for each of the two unmoved beads, and one for each of the identically-colored mirror-image pairs. Thus there are $4!S(5,4) = 240$ invariants for each of these 4 flips.

Using Burnside's lemma, adding up the invariants, in order, of the identity, the odd rotations, the even-but-not-multiple-of-4 rotations, the rotation by 4 beads, the reflections without fixed points, and the reflections with fixed points, we find that:

$$|S/D_8| = \frac{4!S(8,4) + 4 \cdot 4!S(1,4) + 2 \cdot 4!S(2,4) + 4!S(4,4) + 4 \cdot 4!S(4,4) + 4 \cdot 4!S(5,4)}{16}$$

which can be evaluated to be 2619.

Note that a benefit of writing it out in full is that we could easily generalize to find the number of such necklaces using $n$ colors to be:

$$\frac{n!S(8,n) + 4 \cdot n!S(1,n) + 2 \cdot n!S(2,n) + 5 \cdot n!S(4,n) + 4 \cdot n!S(5,n)}{16}$$

3. **(25 points)** *Answer the following questions about icosohedron-coloring.*

   (a) **(5 points)** *Identify the 60 rotation-permutations of the icosohedron. You need not explicitly give all 60; merely give a classification scheme which identifies 60 different rotations.*

Note than an icosohedron has 12 vertices (which can be divided, for rotation purposes, into 6 pairs of opposite vertices), 30 edges (forming 15 pairs of opposite edges), and 20 faces (forming 10 pairs of opposite faces). We thus have 31 axes of rotation to investigate.

Let us consider rotation around a vertex-to-vertex axis. Viewing a vertex straight-on, an icosahedron has 5-fold symmetry. Thus, an axis through the vertex admits 4 rotations: 72°, 144°, 216°, and 288°. These 6 axes can each be rotated in any of these 4 ways, yielding 24 rotations.

Now, looking at an edge-to-edge axis, it is clear the edge (and from there the whole icosahedron) can only be mapped to itself by an 180° rotation. Thus each of these 15 axes only admits a single rotation, describing 15 rotations in all.

Now considering face-to-face axes, if we view an icosohedron face-forward, we observe 3-fold symmetry, so around such an axis, we may rotate 120° or 240°. Thus, these 10 axes allow two rotations each, for a total of 20 rotations.

Finally, there is the identity permutation, described by a 0° rotation about any axis.

Adding all these cases together, there are $24+15+20+1 = 60$ rotions to permute the icosahedron.

(b) **(10 points)** *Using your above rotations and Burnside's lemma, determine how many distinct ways there are to color the 20 faces of the icosohedron with 2 colors if two colorings are regarded as identical if they are rotations of each other. Note: this calculation will include large exponents; you may use a computer to calculate or leave them unreduced. Then, generalize your result to indicate how many ways there are to color the faces of an icosohedron with n colors.*

Here, our underlying set $S$ of colorings not considering symmetry has size $2^{20}$, since, for each face, it can be either of two colors. The identity map thus has $2^{20}$ invariants, since all of $S$ is invariant under the identity.

For any vertex-axis rotation, the faces will be divided into 4 classes which must be the same color: the five faces around the "north pole", the five around the "south pole", the five immediately below the north-polar five, and the five immediately above the south-polar five. Thus, since these face-sets must each be a single color in order to be invariant under such a rotation, we have 4 choices of color to be made, so that every vertex-rotation (of which, as was seen above, there are 24 in total), has $2^4 = 16$ invariants.

For an edge-axis rotation, it can be observed that no face is fixed in place and that pairs of faces swap position under an 180° rotation. Thus, in order to be invariant under an edge-axis rotation, the pairs of faces mapped to each other by the rotation must be the same color, so an invariant under any of the 15 edge-rotations is determined by choosing 10 colors for the 10 pairs of vertices, giving a total of $2^{10} = 1024$ invariants under each of these rotations.

Finally, under a face-axis rotation, the face through which the axis passes is fixed, but all other faces are collected into triples which rotate onto each other. Rotation around a face-axis thus divides the faces into 8 cycles: the north face, the south face, and six triples from among the other 18 faces, which can be visualized on a sample icosohedron. Thus, an invariant under face-axis rotation must have all the faces within a single cycle the same color, so there are $2^8 = 256$ invariants under any of the 20 face-axis rotations.

Assembling these discoveries by way of Burnside's lemma, we see that

$$|S/I| = \frac{2^{20} + 24 \cdot 2^4 + 15 \cdot 2^{10} + 20 \cdot 2^8}{60} = 17824$$

If we were using $n$ colors rather than 2, we would have $n$ choices for each color-class rather than 2, giving the formula

$$|S/I| = \frac{n^{20} + 24 \cdot n^4 + 15 \cdot n^{10} + 20 \cdot n^8}{60}$$

(c) **(10 points)** *How many ways are there to color the* vertices *of an icosohedron with 2 colors? With n colors?*

The icosohedron has 12 vertices, so $|S| = 2^{12}$, which will also be the number of invariants under the identity permutation.

For any of the 24 vertex-axis rotations, 2 vertices will be fixed, while the other 5 vertices in the northern hemisphere will be cyclically mapped onto each other, and likewise for the 5 southern-hemisphere vertices. Thus, an invariant under such a rotation is determined by 4 color selections: the north pole color, the south pole color, the northern hemisphere vertex color, and the southern hemisphere vertex color. There are thus $2^4 = 16$ invariants under these rotations.

The edge-axis rotations map swap pairs of vertices, so the vertices are collected into 6 swapped pairs. To be invariant, each pair must be a single color, so the 15 edge-axis rotations each have $2^6 = 64$ invariants.

The face-axis rotations collect the vertices into cycles of three, so the 12 vertices form 4 cycles. To be invariant under a face-axis rotation, all vertices in a cycle must be the same color, so the 20 face-axis rotations have $2^4 = 16$ invariants.

Using Burnside's Lemma:

$$|S/I| = \frac{2^{12} + 24 \cdot 2^4 + 15 \cdot 2^6 + 20 \cdot 2^4}{60} = 96$$

And, if we were permitted $n$ colors instead of 2:

$$|S/I| = \frac{n^{12} + 24 \cdot n^4 + 15 \cdot n^6 + 20 \cdot n^4}{60}$$

4. **(5 point bonus)** *Let $A$ be a set of $n$-colorings of a $p$-gon, where $p$ is prime, such that if a coloring $X$ is in $A$, so is every rotation of $X$. Recalling that $A/C_p$ is the set of equivalence classes of $A$ under rotation, prove that $\frac{|A|}{p} \leq |A/C_p| \leq \frac{|A|+(p-1)n}{p}$.*

Let us consider a $p$-tuple $\mathbf{x}$ from $\{1, 2, 3, \ldots, n\}^p$ which is invariant under the rotation $r^k$ for $0 < k < p$. We know that $x_a = x_b$ if $b \equiv a + k \pmod{p}$. Applying the chain of such equalities $x_{a_0} = x_{a_1} = x_{a_2} = \cdots = x_{a_\ell}$ where each $a_i \equiv a_{i-1} + k \pmod{p}$, we find that $x_{a_0} = x_{a_\ell}$ if $a_\ell \equiv a_0 + k\ell \pmod{p}$. Since $p$ and $k$ are relatively prime, $k\ell$ can be congruent to any value modulo $p$ as $\ell$ varies. Thus, $\mathbf{x}$ will be invariant under the rotation $r^k$ if and only if $x_1 = x_2 = \cdots = x_p$. There are only $n$ such $p$-tuples in total, and they might not all be in $A$; thus we have demonstrated that, for $0 < k < p$, $0 \leq \mathrm{Inv}(r^k) \leq n$. By Burnside's Lemma, $|A/C_n| = \frac{1}{C_p}\left(\mathrm{Inv}(e) + \sum_{k=1}^{p-1} \mathrm{Inv}(r^k)\right)$; using our known invariant counts of $\mathrm{Inv}(e) = |A|$ and $0 \leq \mathrm{Inv}(r^k) \leq n$, the above becomes:

$$\frac{|A| + 0(p-1)}{p} \leq |A/C_p| \leq \frac{|A| + n(p-1)}{p}$$