

This problem set is due at the beginning of class on *November 10*.

1. **(5 points)** Find an asymptotically accurate approximation for $\binom{n^2}{n}$ in terms of polynomials, exponentials, and self-exponentials. You may write it in big-O notation if you wish.
2. **(10 points)** You have a large supply of beads of 4 different colors and want to string eight of them on a necklace, *making use of each bead at least once*. How many ways are there to do so, if necklaces are considered identical if they are rotations or reflections of each other?
3. **(25 points)** Answer the following questions about icosahedron-coloring. You may find it useful to assemble the model at http://www.korthalsalters.com/model.php?name_en=icosahedron in order to help your visualization.
 - (a) **(5 points)** Identify the 60 rotation-permutations of the icosahedron. You need not explicitly give all 60; merely give a classification scheme which identifies 60 different rotations.
 - (b) **(10 points)** Using your above rotations and Burnside's lemma, determine how many distinct ways there are to color the 20 faces of the icosahedron with 2 colors if two colorings are regarded as identical if they are rotations of each other. Note: this calculation will include large exponents; you may use a computer to calculate or leave them unreduced. Then, generalize your result to indicate how many ways there are to color the faces of an icosahedron with n colors.
 - (c) **(10 points)** How many ways are there to color the *vertices* of an icosahedron with 2 colors? With n colors?
4. **(5 point bonus)** Let A be a set of n -colorings of a p -gon, where p is prime, such that if a coloring X is in A , so is every rotation of X . Recalling that A/C_n is the set of equivalence classes of A under rotation, prove that $\frac{|A|}{p} \leq |A/C_n| \leq \frac{|A|+(p-1)n}{p}$.

I was seriously tormented by the thought of the exhaustibility of musical combinations. The octave consists only of five tones and two semi-tones, which can be put together in only a limited number of ways, of which but a small proportion are beautiful: most of these, it seemed to me, must have been already discovered, and there could not be room for a long succession of Mozarts and Webers, to strike out, as these had done, entirely new and surpassingly rich veins of musical beauty. —John Stuart Mill