1. **(25 points)** *Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, let us consider the set $S \times T$ subjected to the relation $\preceq$ which is such that $(a, b) \preceq (c, d)$ if and only if $a \preceq_S c$ and $b \preceq_S d$.*

   (a) **(5 points)** *Show that $\preceq$ is a partial ordering on $(S \times T)$.*

   For any $s \in S$ and $t \in T$, reflexivity of $\prec_S$ and $\prec_T$ guarantee that $s \preceq_S s$ and $t \preceq_T t$, so $(s, t) \preceq (s, t)$. Thus $\preceq$ is reflexive.

   If (a, b), (c, d), and (e, f) are elements of $S \times T$ such that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, the definition of $\preceq$ guarantees that $a \preceq_S c$ and $c \preceq_S e$, so by transitivity of $\preceq_S$, $a \preceq_S e$; likewise $b \preceq_T d$ and $d \preceq_T f$, so since $\preceq_T$ is transitive, $b \preceq f$. Thus, $(a, b) \preceq (e, f)$, exhibiting transitivity of $\preceq$.

   Finally, let us demonstrate antisymmetry. For (a, b) and (c, d) in $S \times T$, let us explore the circumstances under which it can be true that both $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$. The former relation requires that $a \preceq_S c$ and $b \preceq_T d$; the latter that $c \preceq_S a$ and $d \preceq_T b$. By antisymmetry of $\preceq_S$, since $a \preceq_S c$ and $c \preceq_S a$, it must be the case that $a = c$; likewise, $b \preceq_T d$ and $d \preceq_T d$ requires that $b = d$. Thus, $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$ only if $(a, b) = (c, d)$, so $\preceq$ is antisymmetric.

   (b) **(5 points)** *Show that (a, b) is a maximal (or minimal) element of $S \times T$ if and only if a and b are maximal (or minimal) elements of S and T respectively.*

   We need only prove this for the maximal case; the argument for minimal (a, b) is identical with all orderings reversed.

   If (a, b) is maximal, then for all $(c, d) \neq (a, b)$, it follows that $(a, b) \not\preceq (c, d)$; in particular, for all $d \neq b$, we know that $(a, b) \not\preceq (a, d)$. This will be true only if either $a \not\preceq_S a$ or $b \not\preceq_T d$. The first possibility is clearly not true (by reflexivity of $\not\preceq_S$); thus the latter must be true, and since it is true for all d, b must be maximal in T. A similar inspection of the comparison $(a, b) \not\preceq (c, b)$ will demonstrate maximality of a in S.

   Conversely, if a is maximal in S and b is maximal in T, then we may show that (a, b) is maximal in $S \times T$ by considering the circumstances under which $(a, b) \preceq (c, d)$. This can happen only if $a \preceq c$ and $b \preceq d$; however, by maximality of a and b, this is only the case if $a = c$ and $b = d$. Thus, the only (c, d) for which $(a, b) \leq (c, d)$ is one equal to (a, b), so (a, b) is maximal.

   (c) **(5 points)** *Show that if C is a chain in $S \times T$, then the set C_S of first coordinates of elements of C is a chain in S. (similarly, it is true that the set C_T of second coordinates of elements of C is a chain in T).*

   If x and y are elements of C_S, then by the construction of C_S there must be elements (x, a) and (y, b) of C. Since C is a chain in $S \times T$, it must be the case that $(x, a) \preceq (y, b)$ or $(y, b) \preceq (x, a)$. Thus, by the definition of $\preceq$, either $x \preceq_S y$ or $y \preceq_S x$. Thus x and y are comparable; since x and y were arbitrarily chosen, it follows that any two members of C_S are comparable, so C_S is a chain. The argument for C_T proceeds along similar lines.

   (d) **(5 points)** *Demonstrate that if A is an antichain in $S \times T$, then the sets A_S and A_T defined as in the previous question need not be antichains.*

A simple example suffices to show otherwise: let $S = T = \{1, 2\}$, subject to the ordinary numeric order $\leq$. Then $\{(1, 2), (2, 1)\}$ is an antichain in $S \times T$, but A_S and A_T will both be $\{1, 2\}$, which is not an antichain.

(e) **(5 points + 5 points bonus)** *If S has height h_S and width w_S, and T has height h_T and width w_T, what upper and lower bounds can you put on the height and width of $S \times T$? Can you show that your upper and lower bounds are achieved by some choices of S and T?*

The height of $S \times T$ can be easily seen to be $h_S + h_T - 1$. This can be shown to be a lower bound on the height by explicitly producing a chain of this length. Let $s_1 \preceq_S s_2 \preceq_S s_3 \preceq_S \cdots \preceq_S s_{h_S}$ be a longest chain in S (with each $s_i \neq s_{i+1}$); likewise let $t_1 \preceq_T t_2 \preceq_T t_3 \preceq_T \cdots \preceq_T t_{h_T}$ be a longest chain in T. Then we can construct a chain of length $h_S + h_T - 1$ in $S \times T$ as such:

$$(s_1, t_1) \preceq (s_1, t_2) \preceq \cdots \preceq (s_1, t_{h_T}) \preceq (s_2, t_{h_T}) \preceq \cdots \preceq (s_{h_S}, t_{h_T})$$

Showing that no longer chain is possible is slightly more difficult. Let us suppose C is a chain of length $n = h_S + h_T$ in $S \times T$; we may explicitly label its members in order:

$$(s_1, t_1) \preceq (s_2, t_2) \preceq (s_3, t_3) \preceq \cdots \preceq (s_n, t_n)$$

such that each $(s_i, t_i) \neq (s_{i+1}, t_{i+1})$. By the definition of $\preceq$, the above chain-ordering is identical to asserting the existence of the following *nonstrictly* increasing sequences:

$$s_1 \preceq_S s_2 \preceq_S s_3 \preceq_S \cdots \preceq_S s_n$$

$$t_1 \preceq_T t_2 \preceq_T t_3 \preceq_T \cdots \preceq_T t_n$$

And, since $(s_i, t_i) \neq (s_{i+1}, t_{i+1})$ for all i, it is necessary that for each i, either $s_i \neq s_{i+1}$ or $t_i \neq t_{i+1}$. One question thus to be asked of each of the nonstrictly increasing sequences above is how many of the $\preceq$ symbols actually represent equality. Since the distinct terms of each sequence form a chain, we know that there are at most h_S distinct elements of the first sequence, and at most h_T distinct elements of the second sequence. Thus, there must be at least $n - h_S = h_T$ equalities in the first sequence, and $n - h_T - h_S$ equalities in the second sequence. Thus, of the $2(n-1)$ relation-symbols appearing in those two sequences, at least $h_T + h_S = n$ of them must be equalities; by the Pigeonhole Principle, two of those equalities will be directly atop one another, implying that both $s_i = s_{i+1}$ and $t_i = t_{i+1}$, which contradicts our clain that all elements of C are distinct; thus, no chain of length $h_S + h_T$ exists in $S \times T$, so the height of $S \times T$ is exactly $h_S + h_T - 1$.

Let us now consider antichains. These are much more difficult to craft, in general. We can find a quite good lower bound of $w_S w_T$ for the width: if A_S and A_T are antichains in S and T respectively, it's quite easy to show that $A_S \times A_T$ is an antichain in $S \times T$, so choosing maximum A_S and A_T, this will produce an antichain of size $w_S w_T$ in $S \times T$. An upper bound is somewhat harder. We can invoke Dilworth's Theorem to possibly improve this somewhat, but not much: we know that $|S \times T| \leq h_{S\times T} w_{S\times T}$, so $w_{S\times T} \geq \frac{|S|\cdot|T|}{h_S+h_T-1}$, which may be larger that $w_S w_T$. A good upper bound, however, is very hard to demonstrate. A sloppy easy upper bound would be $\min(w_S|T|+1, w_T|S|+1)$. To show that any subset A of $S \times T$ of size $w_S|T|+1$ contains comparable elements, we may use the Pigeonhole Principle, which says that, since there are only $|T|$ different second coordinates possible, at least $w_S + 1$ elements of A have the same second coordinate. Let us call these elements $(s_1, t), (s_2, t), \ldots, (s_{w_S+1}, t)$. Note that $\{s_1, s_2, \ldots, s_{w_S+1}\}$ is a subset of S of size $w_S + 1$, so by the definition of width, this set can not be an antichain so some $s_i \preceq_S s_j$ in this set. Then, since $t \preceq_T t$, it follows that $(s_i, t) \preceq (s_j, t)$ so A is not an antichain.

2. **(15 points)** *There are 34 non-isomorphic graphs on five vertices. We shall prove this without recourse to brute force.*

   (a) **(5 points)** *There are 120 permutations of five elements. They can be divided into seven categories based on their cycle index. Identify the number of permutations in each category, justifying your work.*

   The seven categories actually correspond exactly to the seven partitions of 5 into positive integers: $1+1+1+1+1$ corresponds to the cycle index x_1^5, which describes the identity (which is unique).

   $2+1+1+1$ corresponds to cycle index $x_2^1 x_1^3$, which is a permutation consisting of three fixed points and a single swap. There are $\binom{5}{2} = 10$ of these, since the permutation is uniquely determined by selecting the elements of the swap.

   $2+2+1$ corresponds to cycle index $x_2^2 x_1^1$, which is a permutation consisting of a fixed point and two swaps. There are $\frac{1}{2}\binom{5}{2,2,1} = 15$ of these, since we may divide the elements of the permutation up among the first swap, second swap, and fixed point, but then must divide by 2 since the "first" and "second" swaps are interchangeable labels.

   $3+1+1$ corresponds to cycle index $x_3^1 x_1^2$, which is a permutation consisting of two fixed points and a 3-cycle. There are $\frac{3!}{3}\binom{5}{3} = 20$ of these, since the permutation is uniquely determined by selecting the elements of the cycle and then their order within the cycle.

   $3+2$ corresponds to cycle index $x_3^1 x_2^1$, which is a permutation consisting of a 3-cycle and a swap. There are $\frac{3!}{3}\binom{5}{3} = 20$ of these, since the permutation is uniquely determined by selecting the elements of the cycle and then their order within the cycle.

   $4+1$ corresponds to cycle index $x_4^1 x_1^1$, which is a permutation consisting of a 4-cycle and a fixed point. There are $\frac{4!}{4}\binom{5}{4} = 30$ of these, since the permutation is uniquely determined by selecting the elements of the cycle and then their order within the cycle.

   5 corresponds to cycle index x_5^1, which is a permutation consisting of one large 5-cycle. There are $\frac{5!}{5} = 24$ of these, since the permutation is uniquely determined by selecting the order of elements within the cycle.

   Note that $1+10+15+20+20+30+24 = 120$, as desired.

(b) **(5 points)** *For each of the seven categories, determine how many graphs are invariant under a representative permutation in that category.*

There are 2^{10} ways to produce a graph on a labeled set of vertices $\{v_1, v_2, v_3, v_4, v_5\}$, since there are 10 pairs of vertices, and for each vertex-pair $[i, j]$, there is the choice of whether to include the edge e_{ij} or not.

All of these graphs are invariant under the identity, so $\text{Inv}(e) = 2^{10}$.

Looking at the single swap, let us consider specifically the swap (v_1v_2). This will swap each of the edges/non-edges e_{13}, e_{14}, and e_{15} with e_{23}, e_{24}, and e_{25} respectively, while keeping every other edge in place. Thus, to craft an invariant graph under this swap, we have 7 decisions to make: to include or not include each of the four fixed edges, and to include or not include each of the three pairs of swapped edges. Thus, $\text{Inv}((v_1v_2)(v_3)(v_4)(v_5)) = 2^7$; by symmetry the same invariant will hold for the other permutations of this type.

Now, considering the double swap, let us consider specifically the swaps (v_1v_2) and (v_3v_4). This will swap each of the edges/non-edges e_{13}, e_{14}, e_{15}, and e_{35} with e_{24}, e_{23}, e_{25}, and e_{45} respectively, while keeping every other edge in place. Thus, to craft an invariant graph under this swap, we have 6 decisions to make: to include or not include each of the two fixed edges, and to include or not include each of the four pairs of swapped edges. Thus, $\text{Inv}((v_1v_2)(v_3v_4)(v_5)) = 2^6$; by symmetry the same invariant will hold for the other permutations of this type.

We may look at the 3-cycle $(v_1v_2v_3)$. This permutation will cycle the edges e_{12}, e_{23}, e_{13} onto each other, and also produce the cycles e_{14}, e_{24}, e_{34} and e_{15}, e_{25}, e_{35}. The edge e_{45} is kept fixed. Thus, to craft an invariant graph under this swap, we have 4 decisions to make: to include or not include e_{45}, and to include or not include each of the three 3-cycles of edges. Thus, $\text{Inv}((v_1v_2v_3)(v_4)(v_5)) = 2^4$; by symmetry the same invariant will hold for the other permutations of this type.

Now, the 3-cycle $(v_1v_2v_3)$ with swap (v_4v_5) is under investigation. This permutation will cycle the edges e_{12}, e_{23}, e_{13} onto each other, and also produces a 6-cycle including all of $e_{14}, e_{24}, e_{34}, e_{15}, e_{25}, e_{35}$. Thus, to craft an invariant graph under this swap, we have 3 decisions to make: to include or not include e_{45}, and the include or not include each of the two cycles of edges. Thus, $\text{Inv}((v_1v_2v_3)(v_4v_5)) = 2^3$; by symmetry the same invariant will hold for the other permutations of this type.

Next we look at the 4-cycle $(v_1v_2v_3v_4)$. This permutation will cycle the edges $e_{12}, e_{23}, e_{34}, e_{41}$ onto each other, as well as swapping e_{13} and e_{24}. It also produces a 4-cycle of $e_{15}, e_{25}, e_{35}, e_{45}$. Thus, to craft an invariant graph under this swap, we have 3 decisions to make; one for each of the three cycles. Thus, $\text{Inv}((v_1v_2v_3v_4)v_5)) = 2^3$; by symmetry the same invariant will hold for the other permutations of this type.

Finally we may look at the 5-cycle $(v_1v_2v_3v_4v_5)$. This permutation will cycle the edges $e_{12}, e_{23}, e_{34}, e_{45}, e_{51}$ onto each other, and cycle $e_{13}, e_{24}, e_{35}, e_{14}, e_{25}$. Thus, to craft an invariant graph under this swap, we have 2 decisions to make since there are two cycles. Thus, $\text{Inv}((v_1v_2v_3v_4)v_5)) = 2^2$; by symmetry the same invariant will hold for the other permutations of this type.

(c) **(5 points)** *Using the above work, determine the number of isomorphism classes of five-vertex graphs.*

With results from the previous two steps, Burnside's Lemma makes this straightforward:

$$|S/S_5| = \frac{2^{10} + 10 \cdot 2^7 + 15 \cdot 2^6 + 20 \cdot 2^4 + 20 \cdot 2^3 + 30 \cdot 2^3 + 24 \cdot 2^2}{120} = 34$$