This problem set is due at the beginning of class on *December 3*.

1. **(25 points)** Given finite posets $(S, \preceq_S)$ and $(T, \preceq_T)$, let us consider the set $S \times T$ subjected to the relation $\preceq$ which is such that $(a, b) \preceq (c, d)$ if and only if $a \preceq_S c$ and $b \preceq_S d$.

   (a) **(5 points)** Show that $\preceq$ is a partial ordering on $(S \times T)$.

   (b) **(5 points)** Show that (a, b) is a maximal (or minimal) element of $S \times T$ if and only if a and b are maximal (or minimal) elements of S and T respectively.

   (c) **(5 points)** Show that if C is a chain in $S \times T$, then the set C_S of first coordinates of elements of C is a chain in S. (similarly, it is true that the set C_T of second coordinates of elements of C is a chain in T).

   (d) **(5 points)** Demonstrate that if A is an antichain in $S \times T$, then the sets A_S and A_T defined as in the previous question need not be antichains.

   (e) **(5 points + 5 points bonus)** If S has height h_S and width w_S, and T has height h_T and width w_T, what upper and lower bounds can you put on the height and width of $S \times T$? Can you show that your upper and lower bounds are achieved by some choices of S and T?

2. **(15 points)** There are 34 non-isomorphic graphs on five vertices. We shall prove this without recourse to brute force.

   (a) **(5 points)** There are 120 permutations of five elements. They can be divided into seven categories based on their cycle index. Identify the number of permutations in each category, justifying your work.

   (b) **(5 points)** For each of the seven categories, determine how many graphs are invariant under a representative permutation in that category.

   (c) **(5 points)** Using the above work, determine the number of isomorphism classes of five-vertex graphs.

> *When it was proclaimed that the Library contained all books, the first impression was one of extravagant happiness. All men felt themselves to be the masters of an intact and secret treasure. There was no personal or world problem whose eloquent solution did not exist in some hexagon.... As was natural, this inordinate hope was followed by an excessive depression. The certitude that some shelf in some hexagon held precious books and that these precious books were inaccessible, seemed almost intolerable. A blasphemous sect suggested that the searches should cease and that all men should juggle letters and symbols until they constructed, by an improbable gift of chance, these canonical books.* —*Jorge Luis Borges*