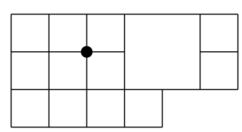
1. **(8 of 12 students attempted this)** *Prove the combinatorial identity* $\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}$. *You may use any method you like.*

   A simple approach is purely combinatorial: the left side of the equation represents the number of ways to choose a subset of size k from an n-element set (which can be done in $\binom{n}{k}$ ways)and then select, with order, an element and then another (possibly identical) element of the subset (which can be done in k^2 ways). These numbers are added up over all values of k, so that the left side represents the number of ways to choose a subset of any size, and then choose two (not necessarily distinct) elements to distinguish individually. In a less formal setting, the objects enumerated by this left side might be considered to be the number of ways to choose a k-person committee from n people, and then to select a chair and secretary (who might be the same person).

   In this context, the right side can be seen to enumerate the same thing through a casewise division: if the chair and secretary are the same, then we could preemptively choose the chair/secretary any of n ways, and then assign committee membership to the remaining $n-1$ prospectives in any of 2^{n-1} ways. On the other hand, if the chair and secretary were different, we would choose any of the n people to be the chair, any of the remaining $n-1$ to be the secretary, and then committ3ee membership for the remaining $n-2$ people could be resolved in any of 2^{n-2} ways.

   There are other approaches possible, using either generating functions or direct algebra, but this is easily the simplest approach.

2. **(12 of 12 students attempted this)** *Answer the following questions relating to paths through the following grid (note the excluded portions):*

   (a) **(5 points)** *How many walks are there from the lower left corner to the upper right corner taking upwards and rightwards steps only?*

   If the grid were complete, there would be $\binom{9}{3}$ routes from the lower left corner $(0,0)$ to the upper right $(6,3)$. We must remove those which pass through the excluded points $(4,2)$ or $(5,0)$ — we don't need to exclude $(6,0)$ since any path through $(6,0)$ would have to go through $(5,0)$ and thus already be excluded.

   Exactly $\binom{6}{2}\binom{5}{1}$ paths pass through $(4,2)$, since there are $\binom{6}{2}$ paths there and $\binom{5}{1}$ paths from there to $(6,3)$. Likewise, exactly $\binom{5}{0}\binom{4}{1}$ paths go through $(5,0)$. No paths go through both points, so re-inclusion of their overlap is not necessary, and we get the result $\binom{9}{3} - \binom{6}{2}\binom{3}{1} - \binom{5}{0}\binom{4}{1} = 35$.

   (b) **(5 points)** *How many of these walks pass through the point marked with a solid dot?*

   If the grid were complete, $\binom{4}{2}\binom{5}{1}$ paths would go through this point. However, we must remove those paths through this point that also go through $(4,2)$ (it is impossible to go through both the solid dot and $(5,0)$, so we don't need to worry about that exclusion. There are $\binom{4}{2}\binom{2}{0}\binom{3}{1}$ paths through both $(2,2)$ and $(4,2)$, so the number of paths through $(2,2)$ not going through $(4,2)$ is $\binom{4}{2}\binom{5}{1} - \binom{4}{2}\binom{2}{0}\binom{3}{1} = 12$, which coluld actually be enumerated through brute force if you were so inclined.

3. **(10 of 12 students attempted this)** *Answer the following questions about the number of solutions a_n to the equation $x_1 + x_2 + x_3 + x_4 = n$ subject to the conditions that all the x_i are non-negative integers, that $x_1 \leq 3$, $2 \leq x_2 \leq 4$, $x_3 \geq 5$, and $x_4 \geq 5$.*

   (a) **(5 points)** *Find a closed form for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.*
   The OGF for selecting x_1 is $1 + x + x^2 + x^3$, or, more compactly, $\frac{1-x^4}{1-x}$. The OGF for selecting x_2 is $x^2 + x^3 + x^4$, or $\frac{x^2-x^5}{1-x}$. The OGF for both x_3 and x_4's selection is $x^5 + x^6 + x^7 + \cdots = \frac{x^5}{1-x}$.
   Thus, the OGF for selecting all of these values is either $\frac{(1-x^4)(x^2-x^5)x^{10}}{(1-x)^4}$ or $\frac{(1+x+x^2+x^3)(x^2+x^3+x^4)x^{10}}{(1-x)^2}$, depending on preference — the first form is much easier to work with in part (b).

   (b) **(5 points)** *Either using your generating function or by other means, find a formula for a_n.*
   Using the generating function:

$$
\begin{aligned}
\frac{(1-x^4)(x^2-x^5)x^{10}}{(1-x)^4} &= \frac{x^{12}-x^{15}-x^{16}-x^{19}}{(1-x)^4} \\
&= (x^{12}-x^{15}-x^{16}+x^{19}) \sum_{n=0}^{\infty} \binom{n+4-1}{4-1} x^n \\
&= \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+12} - \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+15} - \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+16} + \sum_{n=0}^{\infty} \binom{n+}{3} \\
&= \sum_{n=12}^{\infty} \binom{n-9}{3} x^n - \sum_{n=15}^{\infty} \binom{n-12}{3} x^n - 2\sum_{n=16}^{\infty} \binom{n-13}{3} x^n + \sum_{n=19}^{\infty} \binom{n-16}{3} \\
&= \sum_{n=12}^{\infty} \left[\binom{n-9}{3} - \binom{n-12}{3} - \binom{n-13}{3} + \binom{n-16}{3} \right] x^n
\end{aligned}
$$

   so $a_n = \binom{n-9}{3} - \binom{n-12}{3} - \binom{n-13}{3} + \binom{n-16}{3}$.
   This result could also be attained using inclusion-exclusion and pre-emptive assignment of elements: pre-emptively assign a value of 2 to x_2, and 5 to each of x_3 and x_4, leaving $n-12$ to freely distribute; then exclude those cases assigning more than 4 of the remained to x_1, or more than 4 of the remainder to x_2, then re-include the cases doing both.

4. **(6 of 12 students attempted this)** *Prove the combinatorial identity$\sum_{m=1}^{n} \binom{n}{m} S(n-m,k) = (k+1)S(n,k+1)$. You may use any method you like.*

The right side of this equation represents the number of partitions of a n-element set into $k+1$ nonempty sets (which can be done $S(n,k+1)$ ways), and then the selection of one of the parts to be "distinguished". On the left side, we indicate a different procedure for counting the same combinatorial objects: we choose a nonzero size m for the distinguished set, then select a distinguished set of size m in any of $\binom{n}{m}$ ways, and then partition the remaining $n-m$ elements into k undistinguished sets in any of $S(n-m,k)$ ways.

There are certainly other approaches, among them generating-functional and purely algebraic, but this is the most straightforward argument.

5. **(8 of 12 students attempted this)** *You are playing a board game in which on each turn you may advance by crawling one space, walking one space, bouncing two spaces, leaping two spaces, or flying two spaces. Let a_n be the number of different sequences of moves you can use to travel exactly n spaces.*

   (a) **(5 points)** *Derive a recurrence relation and initial conditions for a_n.*

   One can move n spaces by moving $n-2$ spaces (in any of a_{n-2} ways) and then taking either a bounce, leap, or fly; thus, there are $3a_{n-2}$ ways to travel n spaces ending with one of these three moves. Similarly, we could move $n-1$ steps in any of a_{n-1} ways, and then crawl or walk one space, giving $2a_{n-1}$ ways to walk n steps ending with either of these two moves. These exhaust the possible endings, so the total number of sequences traveling n spaces is $3a_{n-2}+2a_{n-1}$. Thus, $a_n = 2a_{n-1}+3a_{n-2}$. The initial conditions are easy: there is only one sequence moving zero steps (the empty sequence), so $a_0 = 1$. There are two sequences moving exactly one space: a single walk or a single crawl. Thus $a_1 = 2$.

   (b) **(5 points)** Solve your recurrence relation to find a closed form for a_n. You may use any method to do so.

   Using classical recurrence-relation solving methods, $a_n = 2a_{n-1}+3a_{n-2}$ has characteristic polynomial x^2-2x-3, which has roots 3 and -1. Thus, $a_n = k\cdot 3^n + \ell(-1)^n$ for some k and ℓ. Plugging in the known values of a_0 and a_1, we get:

$$\begin{aligned} 1 &= k+\ell \\ 2 &= 3k-\ell \end{aligned}$$

   so $k=\frac{3}{4}$ and $\ell=\frac{1}{4}$. Thus,

$$a_n = \frac{3^{n+1}+(-1)^n}{4}$$

6. **(4 of 12 students attempted this)** *Let a_n be the number of strings of length n consisting of the letters A, B, C, and D which contain at least one A and at least one B.*

   (a) *Find a closed form for the exponential generating function $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.*

   The EGF for including one or more "A"s is $x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x-1$; likewise for "B"s. On the other hand, there can be zero or more Cs or Ds, so these have EGF

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$. Thus the EGF for including the appropriate nubmers of any of these letters in a string is the product

$$(e^x - 1)^2(e^x)^2 = e^{4x} - 2e^{3x} + e^{2x}$$

(b) Either using your generating function or by other means, find a formula for a_n.

From the generating function:

$$\begin{aligned} e^{4x} - 2e^{3x} + e^{2x} &= \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - 2\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\ &= \sum_{n=0}^{\infty} (4^n - 2 \cdot 3^n + 2^n) \frac{x^n}{n!} \end{aligned}$$

so $a_n = 4^n - 2 \cdot 3^n + 2^n$. This result could also be obtained by inclusion-exclusion, counting all strings, excluding those omitting either A or B, and re-including those that omit both A and B.