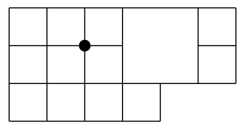
Answer exactly four of the following six questions. *Indicate which four you would like graded!*

Binomial coefficients, Stirling numbers, and arithmetic expressions need not be simplified in your answers.

1. **(10 points)** Prove the combinatorial identity $\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}$. You may use any method you like.

2. **(10 points)** Answer the following questions relating to paths through the following grid (note the excluded portions):



   (a) **(5 points)** How many walks are there from the lower left corner to the upper right corner taking upwards and rightwards steps only?

   (b) **(5 points)** How many of these walks pass through the point marked with a solid dot?

3. **(10 points)** Answer the following questions about the number of solutions a_n to the equation $x_1+x_2+x_3+x_4 = n$ subject to the conditions that all the x_i are non-negative integers, that $x_1 \leq 3$, $2 \leq x_2 \leq 4$, $x_3 \geq 5$, and $x_4 \geq 5$.

   (a) **(5 points)** Find a closed for for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.

   (b) **(5 points)** Either using your generating function or by other means, find a formula for a_n.

4. **(10 points)** Prove the combinatorial identity $\sum_{m=1}^{n} \binom{n}{m} S(n-m,k) = (k+1)S(n,k+1)$. You may use any method you like.

5. **(10 points)** You are playing a board game in which on each turn you may advance by crawling one space, walking one space, bouncing two spaces, leaping two spaces, or flying two spaces. Let a_n be the number of different sequences of moves you can use to travel exactly n spaces.

   (a) **(5 points)** Derive a recurrence relation and initial conditions for a_n.

   (b) **(5 points)** Solve your recurrence relation to find a closed form for a_n. You may use any method to do so.

6. **(10 points)** Let a_n be the number of strings of length n consisting of the letters A, B, C, and D which contain at least one A and at least one B.

   (a) **(5 points)** Find a closed form for the exponential generating function $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

   (b) **(5 points)** Either using your generating function or by other means, find a formula for a_n.