1. **(10 of 12 students attempted this)** *A necklace consists of 6 gems; the gems can be garnets, tourmaline, or zircons. Two necklaces are considered to be identical if one can be obtained by rotating or flipping the other.*

   (a) **(5 points)** *Find a pattern inventory for all such necklaces. You need not algebraically expand the pattern inventory.*

   Let us consider the pattern inventory of invariants under each permutation in $D_6$; we may then use Pólya's Theorem to get the pattern inventory for the equivalnece classes.

   Every sequence of six gems is invariant under the identity permutation $e$; this will yield pattern inventory $(g+t+z)^6$.

   Under the one-element rotation $r$ (or its inverse $r^5$), all gems are in one big cycle, so to be invariant they must all be the same type: our pattern inventory for such invariants is $(g^6+t^6+z^6)$ (which we shall double in our final accounting, since it is the invariant inventory for two different group elements).

   The rotation $r^2$ (and similarly $r^4$ cycles elements in collections of three, yielding pattern inventory $(g^3+t^3+z^3)^2$.

   The rotation $r^3$ swaps opposite pairs, so its invariants have pattern inventory $(g^2+t^2+z^2)^3$.

   Likewise, any of the three flips which move every gem devolve into 3 swaps, with invariant pattern inventory $(g^2+t^2+z^2)^3$.

   The three flips which use an axis passing through two gems, on the other hand, have two fixed points and two swaps, leading to pattern inventory $(g+t+z)^2(g^2+t^2+z^2)^2$.

   Assembling all these in the style of Pólya's theorem, we get:

$$\frac{(g+t+z)^6+2(g^6+t^6+z^6)+2(g^3+t^3+z^3)^2+4(g^2+t^2+z^2)^3+3(g+t+z)^2(g^2+t^2+z^2)^2}{12}$$

   (b) **(5 points)** *Either using your pattern inventory or by other means, determine the number of necklaces which use two of each gemstone.*

   We are looking for coefficients of the form $g^2t^2z^2$. Considering each of the terms in the pattern inventory discovered above in turn, we find that $(g+t+z)^6$ has the term $\binom{6}{2,2,2}g^2t^2z^2$, by the multinomial theorem. Neither $(g^6+t^6+z^6)$ nor $(g^3+t^3+z^3)^2$ have a $g^2t^2z^2$ term. $(g^2+t^2+z^2)^3$ has the term $\binom{3}{1,1,1}g^2t^2z^2$, by the multinomial theorem. Lastly, $(g+t+z)^2(g^2+t^2+z^2)^2$ may also be identified to have the term $\binom{3}{1,1,1}g^2t^2z^2$: after selecting the two $g^2+t^2+z^2$ factors to each contribute one of $g^2$, $t^2$, or $z^2$, there is a unique factor which can be contributed in a unique way by the remaining terms. Thus, the coefficient of $g^2t^2z^2$ in the pattern inventory is

$$\frac{\binom{6}{2,2,2}+2\cdot 0+2\cdot 0+4\binom{3}{1,1,1}+3\binom{3}{1,1,1}}{12}=11$$

2. **(8 of 12 students attempted this)** *A $1 \times n$ checkerboard is to be covered with dominoes (which cover two squares) and checkers (which cover one each). We have dominoes in four colors: green, yellow, purple, and octarine, and checkers in four other colors: black, white, red, and cyan. A checkerboard-covering is called* magical *if it contains exactly one octarine domino, and to the right of the octarine domino uses only checkers.*

   (a) **(5 points)** *Using a casewise analysis on the leftmost object, find a recurrence relation for the number $a_n$ of magical checkerboard-coverings of the $1 \times n$ checkerboard.*

   Let us consider the ways a magical covering of length $n$ could be constructed, from the leftmost object on. If the leftmost object is a checker, then the next $n-1$ spaces will form a magical covering; we can construct such a magical covering in any of $4a_{n-1}$ ways.

   If, on the other hand, the leftmost object is a *non-octarine* domino, then the next $n-2$ spaces will form a magical covering; we can construct such a magical covering in any of $3a_{n-2}$ ways.

   Finally, if the leftmost domino is octarine, then the next $n-2$ spaces must be covered with checkers in order for the coverign to be magical. This can be done in $4^{n-2}$ different ways.

   Thus, $a_n = 4a_{n-1} + 3a_{n-2} + 4^{n-2}$. Our initial conditions will be $a_0 = 0$ and $a_1 = 0$, since a covering of length less than 2 cannot be magical, as it lacks an octarine domino.

   (b) **(5 points)** Using any method or combination of methods you like, solve the recurrence to find a closed form for $a_n$.

   This is a linear nonhomogeneous recurrence relation. We start by finding the general solution of the associated homogeneous recurrence, $b_n = 4b_{n-1} + 3b_{n-2}$, with characteristic polynomial $x^2 - 4x - 3$, which has roots $2 \pm \sqrt{7}$. Thus the general solution for the homogeneous recurrence is $b_n = k(2-\sqrt{7})^n + \ell(2+\sqrt{7})^n$.

   Now, let us find a particular solution to the nonhomogeneous recurrence. The nonhomogeneous part of the recurrence is $4^{n-2}$, or alternatively $\frac{1}{16}4^n$, so the particular solution $a_n^P = C4^n$ seems promising (after verifying that it is not a homogeneous solution). Plugging this into the recurrence, we see that:

$$C4^n = 4 \cdot C4^{n-1} + 3 \cdot C4^{n-2} + 4^{n-2}$$

   which can be solved to give $C = \frac{-1}{3}$, so the general solution to the nonhomogeneous recurrence is

$$a_n = a_n^P + b_n = \frac{-4^n}{3} + k(2-\sqrt{7})^n + \ell(2+\sqrt{7})^n$$

   Plugging in $a_0 = 0$ and $a_1 = 0$, we get the system of equations

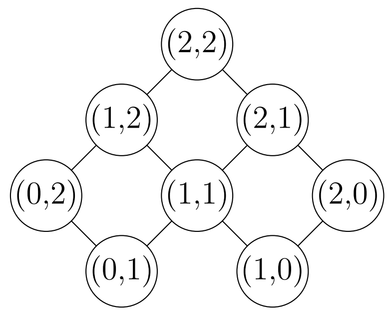
$$\begin{cases} 0 = \dfrac{-1}{3} + k + \ell \\ 0 = \dfrac{-4}{3} + (2-\sqrt{7})k + (2+\sqrt{7})\ell \end{cases}$$

so $k = \frac{7-2\sqrt{7}}{42}$ and $\ell = \frac{7+2\sqrt{7}}{42}$, giving

$$a_n = \frac{-4^n}{2} + \frac{(7-2\sqrt{7})(2-\sqrt{7})^n + (7+2\sqrt{7})(2+\sqrt{7})^n}{42}$$

3. **(11 of 12 students attempted this)** *Let $S$ consist of all ordered pairs with coordinates drawn from the set* $\{0, 1, 2\}$ except for $(0, 0)$ *(so that $|S| = 8$). Consider the ordering $(a, b) \preceq (c, d)$ if $a \leq c$ and $b \leq d$.*

   (a) **(5 points)** *Draw a Hasse diagram for the poset $(S, \preceq)$.*




   (b) **(5 points)** *Identify all the maximal and minimal elements of $(S, \preceq)$. Does $(S, \preceq)$ have a greatest and/or least element? Why or why not?*

   From the Hasse diagram, it can be seen that $(1, 0)$ and $(0, 1)$ are minimal, while $(3, 3)$ is maximal. Since it is the unique maximal element of a finite poset, $(3, 3)$ is also a greatest element; $S$ has no least element, since it has two incomparable minima.

4. **(10 of 12 students attempted this)(10 points)** *Solve the simultaneous recurrence relation:*

$$\begin{cases} a_n = 3a_{n-1} + 2b_{n-1} \\ b_n = \phantom{3}a_{n-1} + 2b_{n-1} \end{cases}$$

*with initial conditions $a_0 = 1$ and $b_0 = 2$. You may use any method you like.*

Here is a solution using ordinary generating functions. Other approaches are also possible, using exponential generating functions, or other forms of vector/matrix representation.

If we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then constructing a power series by adding up each equation above over all values of $n$ above will yield

$$\begin{cases} \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} b_{n-1}x^n \\ \sum_{n=1}^{\infty} b_n x^n = \phantom{3}\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} b_{n-1}x^n \end{cases}$$

which can be simplified to

$$\begin{cases} f(x) - a_0 = 3xf(x) + 2xg(x) \\ g(x) - b_0 = \ \ xf(x) + 2xg(x) \end{cases}$$

so $f(x) = \frac{2xg(x)+1}{1-3x}$ and $g(x) = \frac{xf(x)+2}{1-2x}$. Combining these to isolate $f(x)$, we will get

$$f(x) = \frac{2x\frac{xf(x)+2}{1-2x} + 1}{1-3x} = \frac{2x^2f(x) + 2x + 1}{(1-2x)(1-3x)}$$

so $f(x) = \frac{2x+1}{4x^2-5x+1} = \frac{2x+1}{(1-x)(1-4x)}$. Likewise, $g(x) = \frac{x\frac{2x+1}{(1-x)(1-4x)}+2}{1-2x} = \frac{2-5x}{(1-x)(1-4x)}$.
Using partial fraction decomposition, each of these can be expressed as a sum of the form $\frac{A}{1-x} + \frac{B}{1-4x}$. In particular, $f(x) = \frac{2}{1-4x} - \frac{1}{1-x}$ and $g(x) = \frac{1}{1-4x} + \frac{1}{1-x}$. If we re-express these as power series:

$$\begin{aligned} f(x) &= \frac{2}{1-4x} - \frac{1}{1-x} \\ \sum_{n=0}^{\infty} a_n x^n &= 2\sum_{n=0}^{\infty}(4x)^n - \sum_{n=0}^{\infty} x^n \\ \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty}(2 \cdot 4^n - 1)x^n \end{aligned}$$

so $a_n = 2 \cdot 4^n - 1$. Likewise:

$$\begin{aligned} g(x) &= \frac{1}{1-4x} + \frac{1}{1-x} \\ \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty}(4x)^n + \sum_{n=0}^{\infty} x^n \\ \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty}(\cdot 4^n + 1)x^n \end{aligned}$$

so $b_n = 4^n + 1$.

5. **(7 of 12 students attempted this)** *The following questions pertain to placing 12 chairs around a round table. Two seat-arrangements are considered to be identical if they are rotations of each other.*

   (a) **(5 points)** *Identify the cycle index of every element of the rotation group of a twelve-element set. You may identify a single cycle index as being associated with several different rotations, for brevity.*

   The identity fixes every element, so it has cycle index $x_1^{12}$.

   Since 1, 5, 7, and 11 are relatively prime to 12, the rotations $r$, $r^5$, $r^7$, and $r^{11}$ will actually be 12-element cycles, since no smaller cycles will be induced. These thus have cycle index $x_{12}^1$.

$r^2$ and $r^{10}$ both demonstrably induce two 6-element cycles, so they have cycle index $x_6^2$.

$r^3$ and $r^9$ both demonstrably induce three 4-element cycles, so they have cycle index $x_4^3$.

$r^4$ and $r^8$ both demonstrably induce four 3-element cycles, so they have cycle index $x_3^4$.

Lastly, $r^6$ swaps each of the 6 pairs of opposite seats, giving cycle index $x_2^6$.

The cycle index of the group as a whole would thus be $x_1^{12} + 4x_{12} + 2x_6^2 + 2x_4^3 + 2x_3^4 + x_2^6$. This is not necessary, but will help for the next part of the question.

(b) **(5 points)** *If you have $n$ different styles of chairs, how many distinct seating arrangements are there? You may use every style of chair as many times as you wish.*

Here the underlying set $S$ consists of free choices of $n$ "types" for each of 12 items, so by the specialized version of Burnside's Lemma:

$$|S/C_{12}| = \frac{\mathrm{Cyc}(C_{12})\big|_{x_i=n}}{|C_{12}|} = \frac{x_1^{12} + 4x_{12} + 2x_6^2 + 2x_4^3 + 2x_3^4 + x_2^6|_{x_i=n}}{12}$$
$$= \frac{n^{12} + n^6 + 2n^4 + 2n^3 + 2n^2 + 4n}{12}$$

6. **(2 of 12 students attempted this)** *Let $S_n$ consist of all sequences $1 \le a_1 \le a_2 \le \cdots \le a_n$ such that $a_i \le i$. For instance, $S_3$ consists of the five triples* 111, 112, 113, 122, 123. *Show, either by bijection to a known Catalan-enumerated set, or by appeal to direct enumeration or a recurrence relation, that $|S_n|$ is equal to the Catalan number $C_n$.*

There are several bijections which work well. One simple one is a bijection to the set of parenthesis-nests. Given a parenthesis-nesting, we can craft a sequence by letting $a_i$ be one more than the number of closed parentheses before the $i$th open-parenthesis? Clearly, this forces $a_1 = 1$, since the first open parenthesis has no clsoed parentheses before; likewise, $a_i \le a_{i+1}$, since the number of close-parentheses can only increase as we progress from left to right, and finally, $a_i \le i$, since if we have had only $i$ open parentheses so far, we can have at most $i - 1$ close-parentheses.

We can show this is a bijection by reversing the process. To match a sequence $a_1 a_2 \cdots a_n$ with a parethesis-matching, we may associate $a_1$ with the open-parenthesis ( and each $a_i$ thenceforth with "$)^{a_i - a_{i-1}}($" — that is, with an open parenthesis preceded by $a_i - a_{i-1}$ close-parentheses.