Answer exactly four of the following six questions. *Indicate which four you would like graded!*

Binomial coefficients, Stirling numbers, and arithmetic expressions need not be simplified in your answers.

1. **(10 points)** A necklace consists of 6 gems; the gems can be garnets, tourmaline, or zircons. Two necklaces are considered to be identical if one can be obtained by rotating or flipping the other.

   (a) **(5 points)** Find a pattern inventory for all such necklaces. You need not algebraically expand the pattern inventory.

   (b) **(5 points)** Either using your pattern inventory or by other means, determine the number of necklaces which use two of each gemstone.

2. **(10 points)** A $1 \times n$ checkerboard is to be covered with dominoes (which cover two squares) and checkers (which cover one each). We have dominoes in four colors: green, yellow, purple, and octarine, and checkers in four other colors: black, white, cyan, and red. A checkerboard-covering is called *magical* if it contains exactly one octarine domino, and to the right of the octarine domino uses only checkers.

   (a) **(5 points)** Using a casewise analysis on the leftmost object, find a recurrence relation for the number a_n of magical checkerboard-coverings of the $1 \times n$ checkerboard.

   (b) **(5 points)** Using any method or combination of methods you like, solve the recurrence to find a closed form for a_n.

3. **(10 points)** Let S consist of all ordered pairs with coordinates drawn from the set $\{0, 1, 2\}$ *except for* $(0, 0)$ (so that $|S| = 8$). Consider the ordering $(a, b) \preceq (c, d)$ if $a \leq c$ and $b \leq d$.

   (a) **(5 points)** Draw a Hasse diagram for the poset $(S, \preceq)$.

   (b) **(5 points)** Identify all the maximal and minimal elements of $(S, \preceq)$. Does $(S, \preceq)$ have a greatest and/or least element? Why or why not?

4. **(10 points)** Solve the simultaneous recurrence relation:

$$\begin{cases} a_n = 3a_{n-1} + 2b_{n-1} \\ b_n = a_{n-1} + 2b_{n-1} \end{cases}$$

   with initial conditions $a_0 = 1$ and $b_0 = 2$. You may use any method you like.

5. **(10 points)** The following questions pertain to placing 12 chairs around a round table. Two seat-arrangements are considered to be identical if they are *rotations of each other*.

   (a) **(5 points)** Identify the cycle index of every element of the rotation group of a twelve-element set. You may identify a single cycle index as being associated with several different rotations, for brevity.

(b) **(5 points)** If you have n different styles of chairs, how many distinct seating arrangements are there? You may use every style of chair as many times as you wish.

6. **(10 points)** Let S_n consist of all sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ such that $a_i \leq i$. For instance, S_3 consists of the five triples 111, 112, 113, 122, 123. Show, either by bijection to a known Catalan-enumerated set, or by appeal to direct enumeration or a recurrence relation, that $|S_n|$ is equal to the Catalan number C_n.