Answer exactly four of the following six questions. *Indicate which four you would like graded!*

Binomial coefficients, Stirling numbers, and arithmetic expressions need not be simplified in your answers.

1. **(12 students attempted this problem)** *Answer the following questions about finding the number of words a_n of length n with letters A, B, and C using the letter "A" at least once.*

   (a) **(5 points)** *Find an exponential generating function for a_n.*

   The exponential generating function associated with a string of one or more A's is:

   $$x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = e^x - 1$$

   The exponential generating function associated with a string of zero or more B's (or zero or more C's) is:

   $$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = e^x$$

   Multiplying these three together, a mixed string of one or more A's, zero or more B's, and zero or more C's is associated with the exponential generating function:

   $$(e^x - 1)(e^x)(e^x) = e^{3x} - e^{2x}$$

   (b) **(5 points)** *Develop an argument to show that a_n satisfies the recurrence $a_n = 2a_{n-1} + 3^{n-1}$ with $a_0 = 0$.*

   Clearly $a_0 = 0$, since the unique string of length zero has no "A" in it. We may use a casewise argument on the first character (o last character, if you prefer) in a string of length n.

   If the first character is "B" or "C", than the substring consisting of the $n - 1$ subsequent characters must be a string of A's, B's, and C's which contains at least one A: in other words, a string of length $n - 1$ satisfying our *original* criterion. We know there are a_{n-1} such strings by definition, so there are $2a_{n-1}$ strings of length n beginning with "B" or "C".

   If, on the other hand, the first character is an "A", then the $n - 1$ subsequent characters are no longer restrained — the string as a whole must contain an A, but it is guaranteed to do so! Thus, the following $n - 1$ characters can be any string of "A", "B", and "C", which we know can be produced in 3^{n-1} ways.

   Putting these two cases together, we see that $a_n = 2a_{n-1} + 3^{n-1}$.

   (c) **(5 points)** *Using a method of your choice, find a closed-form expression for a_n.*

   It is easy to derive this from the exponential generating function found in part (a), and only slightly harder to derive it from the recurrence relation found in part (b). There is also a direct enumeration using exclusion methods. Here the EGF approach is demonstrated.

Expanding the exponential generating function we already discovered:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= e^{3x} - e^{2x} \\ &= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\ &= \sum_{n=0}^{\infty} (3^n - 2^n) \frac{x^n}{n!}\end{aligned}$$

so $a_n = 3^n - 2^n$.

2. **(6 students attempted this problem)** *Prove the following identities using combinatorial arguments, where F_n represents the Fibonacci sequence indexed with $F_0 = 1$, $F_1 = 1$, $F_2 = 2$.*

   (a) **(5 points)** $k\binom{n}{k} = n\binom{n-1}{k-1}$.

   There are $\binom{n}{k}$ ways to choose a k-element subset of $\{1, 2, 3, \ldots, n\}$, and k ways to choose a single element of that set, so $k\binom{n}{k}$ enumerates the number of ways to choose a pair (A, x), where $A \subseteq \{1, 2, 3, \ldots, n\}$, $|A| = k$, and $x \in A$. More intuitively, there are $k\binom{n}{k}$ ways to choose a k-person committee and a chair for that committee from a pool of n people.

   We shall see that $n\binom{n-1}{k-1}$ enumerates the same thing by considering a selection procedure where the chair is chosen first, from a pool of n people, and then from the remaining $n - 1$ people, the remaining $k - 1$ normal committee members are selected. More abstractly, there are n choices for x, and then, since we know $x \in A$, our remaining selection is of the $k - 1$ members of $A - \{x\}$.

   (b) **(5 points)** $F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$.

   The Fibonacci number F_n enumerates the coverings of a $1 \times n$ checkerboard by 1×1 checkers and 1×2 dominoes. Let us ask specifically how many such coverings there are using k dominoes. Then, we have $2k$ squares covered by dominoes, and $n - 2k$ squares covered by checkers, so we are laying k dominoes and $n - 2k$ checkers out in a line, which can be done in $\binom{(n-2k)+k}{k} = \binom{n-k}{k}$ ways by selecting the positions in the order for the dominoes. We can use anywhere from zero dominoes to $\lfloor \frac{n}{2} \rfloor$ dominoes, so the number of tilings in total is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$$

   (c) **(5 points)** $\sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} = 3^n$.

   This sum measures the way to select some number (i, but since the sum ranges over all i, we have free choice) of elements form an n-element set, and then further select some subset of these i elements (j of them, specifically, but again, we range over all j). So, the procedure whose possible results this sum enumerates is: the selection of an arbitrary subset A of $\{1, 2, 3, \ldots, n\}$, and then an arbitrary subset B of A. Each number from $\{1, 2, 3, \ldots, n\}$ can thus be assigned one of 3 roles: it can be outside of A, it can be in A but outside B, or it can be in B (and thus necessarily in A). Thus, this sum is 3^n.

3. **(5 students attempted this problem)** *Let* $S = \{2, 3, 4, 5, 6, 7, 9, 15, 30, 210\}$*, and let* $a \preceq b$ *if* b *is divisible by* a*.*

   (a) **(5 points)** *Draw a Hasse diagram for the poset* $(S, \preceq)$*. What are the maximal and minimal elements of this poset? Does* S *have greatest and least elements, and if so, what are they?*

   The maximal elements are , 4, 210 and 9; none of them is the greatest since they are non-unique. The minimal elements are 2, 3, 5, and 7, and again, there is no least since they are non-unique.

   (b) **(5 points)** *State Dilworth's Theorem, and demonstrate explicitly that Dilworth's Theorem is true on the above-mentioned* S*.*

   Dilworth's Theorem: for a finite poset $(S, \preceq)$ of width k, there is a partition of S into k chains. There are many other valid ways to state the same fact.

   The above poset has width 5, with a representative antichain being $\{4, 5, 6, 7, 9\}$. A sample decomposition into 5 chains is $\{2, 4\}$, $\{3, 6, 30, 210\}$, $\{5, 15\}$, $\{7\}$, and $\{9\}$.

   (c) **(5 points)** *Prove that a sequence* $(a_1, a_2, \ldots, a_{n^2+1})$ *of real numbers has either a nonstrictly increasing or nonstrictly decreasing subsequence of length* $n + 1$*.*

   Consider the poset ordering on $\{1, 2, \ldots, n^2 + 1\}$ where $i \preceq j$ if $i \leq j$ and $a_i \leq a_j$. We shall show that every chain of size k indexes a nonstrictly increasing sequence of length n, and every antichain of size k indexes a nonstrictly decreasing sequence.

   Consider a chain C, consisting of distinct $i_1 \preceq i_2 \preceq i_3 \preceq \cdots \preceq i_k$. By the definition of $\preceq$, we know that $i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_k$, and by distinctness, we know that these inequalities are actually strict: $i_1 < i_2 < i_3 < \cdots < i_k$. We also know from the definition of $\preceq$ that $a_{i_1} \leq a_{i_2} \leq a_{i_3} \leq \cdots \leq a_{i_k}$. Since the i_j are themselves shown to be increasing indices, this nonstrictly increasing sequence of a_{i_j} is a subsequence of our original sequence. Thus, every chain of size k indexes a length-k nonstrictly increasing subsequence of $(a_1, a_2, \ldots, a_{n^2+1})$.

   On the other hand, let us now consider an antichain of size k, which we call $A = \{i_1, i_2, \ldots, i_k\}$. We may choose the labels such that $i_1 < i_2 < i_3 < \cdots < i_k$ in the standard integer ordering. However, by incomparability, we know that, in our poset ordering, $i_j \not\preceq i_{j+1}$, so it is not the case that both $i_j \leq i_{j+1}$ and $a_{i_j} \leq a_{i_{j+1}}$. The first condition, however is true by our construction, so the second must be false, and thus $a_{i_j} > a_{i_{j+1}}$. We thus have that $a_{i_1} > a_{i_2} > a_{i_3} > \cdots > a_{i_k}$, so, since the i_j were themselves increasing, this strictly decreasing sequence of a_{i_j} is a subsequence of our original sequence. Thus, every antichain of size k indexes a length-k decreasing subsequence of $(a_1, a_2, \ldots, a_{n^2+1})$ — strictly decreasing in fact, although that is a stronger fact than we need.

Dilworth's Theorem guarantees that the product of the height and width of a poset is at least equal to the poset's cardinality; since the poset described here has n^2+1 elements, it is easy to show that the poset must have either height or width $n+1$, leading to the desired result.

4. **(9 students attempted this problem)** *A completed tic-tac-toe board is a* 3×3 *grid in which each cell contains an "X" or an "O". Two boards are considered to be identical if one can be converted into the other via rotations or reflections.*

   (a) **(5 points)** *Find the number of distinct completed tic-tac-toe boards. You need not arithmetically simplify your result.*

   One can fill a tic-tac-toe board, if we don't take symmetry into consideration, in any of 2^9 ways. This shall form a set of completed boards whose symmetries we will winnow away by applying Burnside's Lemma to the dihedral group D_4.

   Enumerating invariants: we find that under the identity all boards are invariant, for a total of 2^9. Under a 90° rotation clockwise or counterclockwise, we find that, in order to be invariant, all 4 corners must have the same symbol, and all 4 sides the same symbol, while any symbol can be in the middle, which allows 2^3 possibilities. Under a 180° rotation, the opposite corners and sides must have the same symbol to be invariant; this means we have 5 symbols to select (the NW/SE corners; the NE/SW corners; the north/south edges, the east/west edges; and the center) and have 2^5 invariants. Under a horizontal or vertical flip, or even a diagonal flip, as luck would have it, we see the three symbols on opposite sides of the axis mapped onto each other while the three on the axis stay in place, so an invariant here would require selection of 2^6 symbols.

   Thus, using Burnside's Lemma, the number of distinct boards is

$$\frac{2^9 + 2 \cdot 2^3 + 2^5 + 4 \cdot 2^6}{8} = 102$$

   (b) **(5 points)** *Find a pattern inventory for the completed tic-tac-toe boards. You need not algebraically expand your result.*

   We proceed as above, deriving pattern inventories for invariants. The identity-invariants have pattern inventory $(x+o)^9$; the 90° rotations have $(x^4+o^4)^2(x+o)$; the 180° rotation has $(x^2+o^2)^4(x+o)$, and all four flips have $(x^2+o^2)^3(x+o)^3$, giving pattern inventory

$$\frac{(x+o)^9 + 2(x^4+o^4)^2(x+o) + (x^2+o^2)^4(x+o) + 4(x^2+o^2)^3(x+o)^3}{8}$$

   (c) **(5 points)** *When actually playing tic-tac-toe (if we ignore winning conditions), a completed board will have 5 O's and 4 X's. How many distinct boards are there with this distribution of symbols? You need not arithmetically simplify your result.*

   From the pattern inventory developed in part (b), we now wish to extract the xo^5 coefficient. Term-by-term this is mostly not very difficult.

$(x+o)^9$ we know contributes a $\binom{9}{4}x^4o^5$ term, by the binomial theorem; $(x^4+o^4)^2(x+o)$ can only contribute a x^4o^5 by distribution of an x^4 and o^4 from the first two terms in the product, and o from the last, which can occur in only 2 ways. $(x^2+o^2)^4(x+o)$ must get an o from the last term in the product and two each of x^2 and o^2 from the first four terms, which can happen in $\binom{4}{2}$ ways. Lastly, $(x^2+o^2)^3(x+o)^3$ can produce x^4o^5 in a number of ways: we can get o^3 from the last three terms in one way and x^4o^2 from the first in any of $\binom{3}{1}$ ways, or we can get x^2o from the last three terms in $\binom{3}{1}$ ways and x^2o^4 from the first in any of $\binom{3}{1}$ ways. Thus, our total collection of x^4o^5 terms is:

$$\frac{\binom{9}{4}x^4o^5 + 2\cdot 2x^4o^5 + \binom{4}{2}x^4o^5 + 4\left(1\binom{3}{1} + \binom{3}{1}\binom{3}{1}\right)x^4o^5}{8} = 23$$

5. **(11 students attempted this problem)** *We have n undistinguishable identical coins to be distributed among four friends. Attila must receive at least 4 coins. Borbála must be given no more than 3. Csilla can get any number, and Dezső gets either 1 or 2. Let a_n represent the number of ways to distribute the coins.*

   (a) **(5 points)** *Find a closed form for the ordinary generating function of a_n.*

   Attila must receive 4 or more coins, so his associated generating function is $x^4+x^5+x^6+\cdots = \frac{x^4}{1-x}$. Borbála gets no more than 3, yielding the polynomial generating function $1+x+x^2+x^3$; this can be left as is, or converted to the fraction $\frac{1-x^4}{1-x}$. Csilla can have any number, so her representative generating function is $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$. Lastly, Dezső's is the simple $x+x^2$, or alternatively $\frac{x-x^3}{1-x}$ (there may be no benefit in writing this as a fraction, but it's permitted. All together, then, we have the generating function

$$\frac{x^4(1-x^4)(x-x^3)}{(1-x)^4} = \frac{x^5-x^7-x^9+x^{11}}{(1-x)^4}$$

   (b) **(5 points)** *Find a closed form for a_n itself, using any method you like.*

   The easiest approach in light of the work done in part (a) is with ordinary generating function manipulation. An inclusion-exclusion approach is also possible but a bit messier.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \frac{x^5-x^7-x^9+x^{11}}{(1-x)^4} \\
&= (x^5-x^7-x^9+x^{11})\sum_{n=0}^{\infty}\binom{n+3}{3}x^n \\
&= \sum_{n=0}^{\infty}\binom{n+3}{3}x^{n+5} - \sum_{n=0}^{\infty}\binom{n+3}{3}x^{n+7} - \sum_{n=0}^{\infty}\binom{n+3}{3}x^{n+9} + \sum_{n=0}^{\infty}\binom{n-3}{3}x^{n+11} \\
&= \sum_{n=4}^{\infty}\binom{n-2}{3}x^n - \sum_{n=7}^{\infty}\binom{n-4}{3}x^n - \sum_{n=8}^{\infty}\binom{n-6}{3}x^n + \sum_{n=11}^{\infty}\binom{n-8}{3}x^n \\
&= \sum_{n=0}^{\infty}\left[\binom{n-2}{3} - \binom{n-4}{3} - \binom{n-6}{3} + \binom{n-8}{3}\right]x^n
\end{aligned}$$

so $a_n = \binom{n-2}{3} - \binom{n-4}{3} - \binom{n-6}{3} + \binom{n-8}{3}$. This is in fact equal to $8n - 48$ for $n \geq 8$, as some astute students observed.

(c) **(5 points)** *Suppose Enikő also joins the group, and she must be given an even number of coins. Let b_n be the number of ways to distribute the coins among all five friends. What is the closed form for the ordinary generating function of b_n?*

Enikő has an associated generating function of $1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$, giving an overall generating function of

$$\frac{x^4 - x^7 - x^8 + x^{11}}{(1-x)^4(1-x^2)}$$

6. **(5 students attempted this problem)** *Answer the following enumerative questions:*

(a) **(5 points)** *Let a_n be the number of ways to express n as an ordered sum of nonzero integers; e.g. $a_4 = 8$ because*

$$4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$$

Prove that $a_n = 2^{n-1}$.

An ordered partition of n can be associated with a way to lay dividers among n objects lying in a line; between any two elements we can include a divider or not; there are $n-1$ gaps, and thus 2^{n-1} ways to insert dividers.

There are other approaches, such as providing an argument for the recurrance $a_n = 2a_{n-1}$, but this is the most visual.

(b) **(5 points)** *How many* surjective *functions are there from the set $\{1,2,3,4,5,6,7\}$ to the set $\{A,B,C\}$?*

This is definitionally $3!S(7,3)$, or it can be determined by inclusion-exclusion to be

$$3^7 - \binom{3}{1}2^7 + \binom{3}{2}1^7 = 1806$$

(c) **(5 points)** *How many ways are there to seat 5 people at a round table when you have 7 friends to choose from and rotations of a seating arrangement are considered to be identical?*

There are $\frac{7!}{(7-5)!}$ ways to seat people ignoring rotations. Every rotation of an individual setting is distinct, so they can be collected into groups of 5 which are identical under rotation. Thus there are $\frac{7!}{5\cdot 2!} = 504$ seatings.