Answer exactly four of the following six questions. *Indicate which four you would like graded!*

Binomial coefficients, Stirling numbers, and arithmetic expressions need not be simplified in your answers.

1. **(15 points)** Answer the following questions about finding the number of words a_n of length n with letters A, B, and C using the letter "A" *at least once*.
   (a) **(5 points)** Find an exponential generating function for a_n.
   (b) **(5 points)** Develop an argument to show that a_n satisfies the recurrence $a_n = 2a_{n-1} + 3^{n-1}$ with $a_0 = 0$.
   (c) **(5 points)** Using a method of your choice, find a closed-form expression for a_n.

2. **(15 points)** Prove the following identities using combinatorial arguments, where F_n represents the Fibonacci sequence indexed with $F_0 = 1$, $F_1 = 1$, $F_2 = 2$.
   (a) **(5 points)** $k\binom{n}{k} = n\binom{n-1}{k-1}$.
   (b) **(5 points)** $F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$.
   (c) **(5 points)** $\sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} = 3^n$.

3. **(15 points)** Let $S = \{2, 3, 4, 5, 6, 7, 9, 15, 30, 210\}$, and let $a \preceq b$ if b is divisible by a.
   (a) **(5 points)** Draw a Hasse diagram for the poset $(S, \preceq)$. What are the maximal and minimal elements of this poset? Does S have greatest and least elements, and if so, what are they?
   (b) **(5 points)** State Dilworth's Theorem, and demonstrate explicitly that Dilworth's Theorem is true on the above-mentioned S.
   (c) **(5 points)** Prove that a sequence $(a_1, a_2, \ldots, a_{n^2+1})$ of real numbers has either a nonstrictly increasing or nonstrictly decreasing sequence of length $n + 1$. (Hint: consider the poset ordering on $\{1, 2, \ldots, n^2 + 1\}$ where $i \preceq j$ if $i \leq j$ and $a_i \leq j$).

4. **(15 points)** A *completed tic-tac-toe board* is a 3×3 grid in which each cell contains an "X" or an "O". Two boards are considered to be identical if one can be converted into the other via rotations or reflections.
   (a) **(5 points)** Find the number of distinct completed tic-tac-toe boards. You need not arithmetically simplify your result.
   (b) **(5 points)** Find a pattern inventory for the completed tic-tac-toe boards. You need not algebraically expand your result.
   (c) **(5 points)** When actually playing tic-tac-toe (if we ignore winning conditions), a completed board will have 5 O's and 4 X's. How many distinct boards are there with this distribution of symbols? You need not arithmetically simplify your result.

5. We have n undistinguishable identical coins to be distributed among four friends. Attila must receive at least 4 coins. Borbála must be given no more than 3. Csilla can get any number, and Dezső gets either 1 or 2. Let a_n represent the number of ways to distribute the coins.

   (a) **(5 points)** Find a closed form for the ordinary generating function of a_n.

   (b) **(5 points)** Find a closed form for a_n itself, using any method you like.

   (c) **(5 points)** Suppose Enikő also joins the group, nad she must be given an even number of coins. Let b_n be the nubmer of ways to distribute the coins among all five friends. What is the closed form for the ordinary generating function of b_n?

6. **(15 points)** Answer the following enumerative questions:

   (a) **(5 points)** Let a_n be the number of ways to express n as an *ordered* sum of nonzero integers; e.g. $a_4 = 8$ because

   $$4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$$

   Prove that $a_n = 2^{n-1}$.

   (b) **(5 points)** How many *surjective* functions are there from the set $\{1, 2, 3, 4, 5, 6, 7\}$ to the set $\{A, B, C\}$?

   (c) **(5 points)** How many ways are there to seat 5 people at a round table when you have 7 friends to choose from and rotations of a seating arrangement are considered to be identical?