1 What is combinatorics?

Combinatorics is the branch of mathematics dealing with things that are *discrete*, such as the integers, or words created from an alphabet. This is in contrast to analysis, which deals with the properties of *continuous* systems, such as the real number line or a differential equation. Many (but not all) combinatorial problems also address systems which are *finite*, so the systems being investigated will have a specific number of elements: in fact, the question "how many elements are in a particular system?" is a common combinatorial query.

The above loose definition of combinatorics is inclusive of a few other disciplines. Explorations of the arithmetic properties of the integers may be considered to fall under the heading of *number theory*; explorations of multiplication-like and addition-like operations on discrete structures generally falls under the heading of *algebra*. The distinctions between these disciplines and combinatorics, especially as regards finite structures, are somewhat amorphous.

Broadly speaking, combinatorics tends to ask four types of questions:

Enumerative questions How large is a particular set?

Existential questions Is a set of conditions on a discrete system actually satisfiable?

Extremal questions How large does a discrete structure need to become before certain substructures become inevitable?

Computational questions How long would it take a computer to answer a question posed about a discrete system?

To give some context to the above descriptions, here are a few fundamentally combinatorial questions:

- How many numbers less than 10000 consist of distinct digits appearing in ascending order?
- How many people need to be attending a party in order to guarantee that there are either 5 mutual acquaintances or 5 mutual strangers?
- If 20 people each choose a set of acceptable roommates, can each person be paired with an acceptable roommate? How could a computer program be written to find such a pairing?
- How many anagrams are there of the word "MISSISSIPPI"?
- If we remove one square from a 7×9 checkerboard, can the remaining squares be covered with dominoes? Does it matter which square we remove?
- If we walk five blocks north and six blocks east, taking a random route, what is the probability that we will pass by a newsstand one block to the north and one block east of our starting position?

After we have completed both semester of this course, you will know the answer to five of these questions — as well as many others!

2 Introductory Enumerative Methods

The absolute simplest enumerative method is *exhaustive enumeration*, also known as "brute force": simply list out every

Question 1: *How many three-digit numbers are there whose digits add up to 6?*

Answer 1: *Consider those numers starting with "1" and with the remaining digits adding up to 5: 105, 114, 123, 132, 141, and 150. Then those starting with "2", and having remaining digits adding up to 4: 204, 213, 222, 231, 240; now those starting with "3": 303, 312, 321, 330. Finally, "4" yields 402, 411, and 420, "5" gives 501 and 510, and "6" gives only 600. We have listed 21 numbers here.*

Exhaustive enumeration works, if the set being explored is small enough and the approach is methodical enough. But it's easy to get wrong if you're careless. It's better to use principles to build up a count on large sets from small sets.

2.1 A multiplicative principle

If the items being counted can be split into two or more independently countable parts, the *multiplicative principle* is a fundamental means for counting the items in question:

Proposition 1. *If the elements of a set X can be decomposed into pairs of elements of A and B such that each pair (a, b) corresponds to one and only one element of X, then $|X| = |A| \cdot |B|$.*

So, for example: **Question 2:** *How many 2-letter strings can be constructed consisting of a consonant followed by a vowel?*

Answer 2: *There are 21 consonants and 5 vowels. If we were to take A above to be the consonant-set and B the vowel-set, there is a trivial association with our set of two-letter strings via concatenation, e.g. "TO"⇔("T","O"), so the size of our desired set is* $21 \cdot 5 = 105$.

We can also use this to string together groups of more than two items at a time:

Question 3: *A license plate number consists of three letters followed by 3 numbers. To avoid confusion, the letters I and O are not used, as are the numbers 1 and 0. How many license plates are possible?*

Answer 3: *We have 24 possibilities for each letter, and 8 possibilities for each number. Thus, the number of license plates is the product of our number of choices for each position:* $24 \cdot 24 \cdot 24 \cdot 8 \cdot 8 \cdot 8 = (24^3)(8^3) = 7077888$.

As long as the *number* of choices for each part of the problem is independent of the choice made for the other parts of the problem, the multiplicative principle holds, which allows us to use it even in cases where there is apparently not independence of choice:

Question 4: *How many 2-digit numbers are there with distinct digits?*

Answer 4: *The first digit can be any number from 1 through 9; there are 9 possibilities for this digit. It seems that the possibilities for the second digit depend on the choice of the first digit, but regardless of what first digit was chosen, there are 9 possibilities for the second digit (what they are depends on the first digit, but how many of them there are is not). Thus, there are* $9 \cdot 9 = 81$ *such numbers.*

Question 5: *How many orderings are there of the letters "ABCDE"?*

Answer 5: *We can choose any of the 5 letters to be first; we can choose any of the 4 remaining letters to be second; any of the 3 remaining letters to be third, and any of the 2 remaining letters to be fourth. At this point, we have only one letter left, and it must be fifth. Multiplying together the number of choices we have at each phase of the string construction, we have* $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Beware, however, of situations where the number of possibilities for different parts of the problem are not independent!

Question 6: *How many two-letter strings are there with at most one vowel?*

Answer 6: *This is not something we can solve easily using the multiplicative principle! We might naïvely assert that there are 26 possibilities for the first letter, but we run into a problem: if the first letter is a consonant, we have 26 possibilities for the second letter; if it's a vowel, however, the second letter would have to be a consonant, so we'd only have 21 choices.*

2.2 An additive principle

If we can divide the set of objects being counted into two subsets, and count each one individually, we can add up their sizes to find the size of the original set.

Proposition 2. *If set* X *can be partitioned into disjoint subsets* A *and* B*, then* $|X| = |A| + |B|$.

This is useful for problems which behave differently in different cases:

Question 7: *How many two-letter strings are there with at most one vowel?*

Answer 7: *As mentioned above, our choice of the first letter dictates the number of possibilities for the second letter. We can thus divide the set we mean to count into two sets corresponding to our two cases:*

Case I: *The first letter is a vowel (so the second letter must be a consonant). This case allows 5 choices for the first letter and 21 for the second, yielding* $5 \cdot 21 = 105$ *two-letter strings.*

Case II: *The first letter is a consonant (so the second letter could be anything). This case allows 21 choices for the first letter, and 26 for the second, yielding* $21 \cdot 26 = 546$ *two-letter strings.*

We add these two distinct cases to get the total number of strings, $105 + 546 = 651$.

As seen above, the additive principle is frequently used in tandem with other methods. In addition, it can be used in place of other methods, if necessary:

Question 8: *How many 2-letter strings can be constructed consisting of either a consonant followed by a vowel, or a vowel followed by a consonant?*

Answer 8: *We can divide this into two cases: consonant followed by a vowel has* $21 \cdot 5 = 105$ *possibilities, and vowel followed by a conosonant has* $5 \cdot 21 = 105$ *possibilities, for a total of* $105 + 105 = 210$ *possibilities.*

However, we could also solve this using solely the multiplicative principle, by considering three things to be set: a choice of vowel, a choice of consonant, and an order, e.g. "TO"⇔("O","T",C–V), "OF"⇔("O","F",V–C). There are 5 choices of vowel, 21 choices of consonant, and two orders, giving $5 \cdot 21 \cdot 2 = 210$ *possibilities.*

2.3 A Subtractive Principle

Sometimes, it's easier to overcount a set and then exclude the things you didn't mean to count than it is to try to exclude them from the start.

Proposition 3. *If set X is a subset of set A, and B consists of all elements of A not in X, then $|X| = |A| - |B|$.*

Question 9: *How many two-letter strings are there with at most one vowel? (haven't we already asked this one?)*

Answer 9: *We can count all two-letter strings easily: there are 26 choices of first letter, and 26 choices of second letter, for $26 \cdot 26 = 676$. But this includes two-letter strings with two vowels, and we emphatically don't want to include those. How many of them are there? Well, we'd need two vowels, so there would be $5 \cdot 5 = 25$ ways to get these unwanted strings. There are thus $676 - 25 = 651$ strings with fewer than 2 vowels.*

The subtractive principle as presented here is actually not of use too often, but some modifications on the same idea we'll find to be extremely useful later.

2.4 Symmetry removal with division

This principle is extremely useful, but also dangerously easy to misuse:

Proposition 4. *If each element of set X can be associated with n elements of set A such that each element of set A is associated with exactly one element of X, $|X| = \frac{|A|}{n}$.*

The use of this method can be seen with a simple enumeration which we will find extremely useful later:

Question 10: *How many 3-element subsets are there of the set $\{0, 1, 2, \ldots, 9\}$?*

Answer 10: *We will start with a question akin to one we've already answered: how many sequences of three distinct digits from 0 to 9 are there? There are 10 possibilities for the first digit, and any of the nine unused possibilities can be used for the second digit, and any of the eight remaining after that can be used for the third. So there are $10 \cdot 9 \cdot 8 = 720$ possible sequences.*

But sequences are not sets! The sequence 314 and 134 both correspond to the set $\{1, 3, 4\}$. However, sequences of distinct terms correspond to sets in a nice predictable fashion: each set of three elements can have any of the 3 elements first, any of the 2 remaining elements second, and the leftover element third, so each set corresponds to 6 different sequences when reordered. Thus, to find the number of sets, we simply count "groups of 6 sequences", which would be $\frac{720}{6} = 120$.

Be wary, however, of situations where a correspondence is not consistently n-to-one for a particular value of n:

Question 11: *How many choices of 3 numbers from the set $\{0, 1, 2, \ldots, 9\}$, allowing repetitions and ignoring order of selection, are there?*

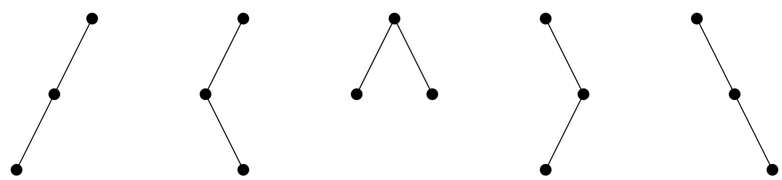
Answer 11: *This seems quite similar to the above problem: why not count strings of 3 numbers and divide by the number of arrangements which correspond to the same thing in different orders? But different choices correspond to different nubmers of strings. While $[1, 3, 4]$ corresponds to 6 strigns as seen above, $[1, 1, 4]$ only corresponds to 3 (114, 141, and 411), and $[1, 1, 1]$ only corresponds to one.*

You can use the division method on this if you use the additive principle to partition the set of choices into three classes based on how many of their elements are the same: $[a, a, a]$*,* $[a, b, b]$*, and* $[a, b, c]$*. It is left as an exercise to the reader to get from this method to the correct answer, which is* 220*.*

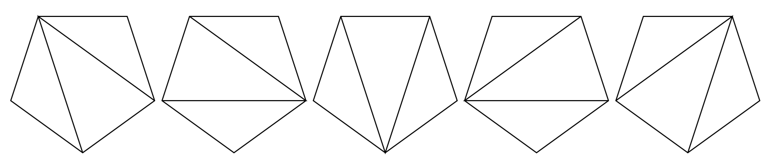
3 Two Forms of Equivalence

3.1 Bijections Between Countable Structures

One useful trick which we'll see more in coming days is showing that the same enumeration can be used to count different structures. For instance, we might consider the ways to build a binary tree with 3 nodes; there are 5 ways to do this:



We might also consider the numer of ways to triangulate a pentagon:



Another question whose answer is 5 is "How many ways are there to nest 3 pairs of parentheses". We can do this in the following ways: ((())), (()()), (())(), ()(()), and ()()(). These might be coincidence! But if we look at 4-node binary trees, triangulations of the hexagon, or nestings of 4 sets of parentheses, we see that there are 14 ways to do each of these. This may be more than coincidence! In fact, we may show that all of these objects are countable the same way by explicitly constructing a bijection between them. For instance, we can turn a parenthesis structure into a tree by the rule: "$(A)B$" becomes a node whose left child is whatever tree is associated with "A" and whose right child is whatever tree is associated with "B"; conversely, we can associate any tree with a parenthesis-system by associating a tree with left-branch "A" and right-branch "B" with the parenthesis-system $(A)B$. Similarly, triangulations could be associated with trees by identifying each trangle with a node, which produces a natural binary tree. By associating these systems with each other we show that they are enumerated the same way. This is useful in case one of these structures is easier to count than the others (in this case, binary trees are particularly easy to count). The numbers in this example are called the *Catalan numbers*. Richard Stanley, in *Enumerative Combinatorics, Volume 2*, identifies 66 different structures enumerated by the Catalan numbers.

3.2 Multiple Counting Methodologies For a Single Structure

In addition to using one counting method to enumerate multiple different-seeming structure, one powerful trick in our arsenal is using two different methods to count the same thing, and in so doing build an equality between two different-seeming algebraic quantities.

Example: Question 7 and Question 9 really asked the exact same thing, but we answered it two different ways. From this, even if we couldn't do simple arithmetic, we'd know that $5 \cdot 21 + 21 \cdot 26 = 26 \cdot 26 - 5 \cdot 5$.

That's a pretty silly identity, since it's just arithmetic. However, we could use the same technique, slightly generalized, to prove a familiar algebraic identity:

Question 12: *Give a combinatorial proof that for integers $m \geq n \geq 0$, $m^2 - n^2 = (m+n)(m-n)$.*

Answer 12: *Let $X = \{a_1, a_2, \ldots, a_n, b_{n+1}, b_{n+2}, \ldots, b_m\}$. We will ask the question "how many ways are there to select an ordered pair of elements of X such that not both of them are as?" This is a generalization of our above argument, with a_i and b_j as standins for vowels and consonants respectively.*

We may count the number of ways to get such an ordered pair using the additive principle. If the first term in the ordered pair is an a_i (which could happen in any of n ways), then the second term must be a b_j (which can happen in any of $m - n$ ways). We have $n(m - n)$ possible ordered pairs beginning with an a. If we start with a b, however (whcih can be done in any of $m - n$ ways), we have free choice of any of the m elements of X for the second term. Thus there are $(m - n)m$ ordered pairs whose first term is a b. Taking these possibilities together, we see that there are $n(n - m) + (m - n)m = (n - m)(n + m)$ ordered pairs in total.

Alternatively, however, we could count all the ordered pairs (which have m choices for each term, for a total of m^2) and use the subtractive principle to remove those of the form (a_i, a_j), of which there can be easily seen to be n^2. Thus, this same enumeration can be computed as $m^2 - n^2$. We thus have a combinatorial presentation that $(n - m)(n - m) = m^2 - n^2$.

This too is a bit of a parlor trick: it avoids using the distributive property, but it's otherwise difficult to see why this might be useful. The big benefits of demonstrating equality this way came into play when we introduce the combination statistic.

Definition 1. The number of possible k-element subsets of an n-element set is the *combination statistic* or *binomial coefficient*, denoted $\binom{n}{k}$, and read "n choose k". This expression is equal to $\frac{n!}{k!(n-k)!}$.

The demonstration that this statistic is actually $\frac{n!}{k!(n-k)!}$ is simply a generalization of the argument used to solve Question 11.

The binomial coefficients are subject to a tremendous number of identities; one of the simplest is $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

This can be proved algebraically, but doing so is perfectly miserable, and more importantly, unilluminating:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!(n-k)}{k![(n-k)(n-k-1)!]} + \frac{(n-1)!(k)}{[k(k-1)!](n-k)!} \\
&= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!(k)}{k!(n-k)!} \\
&= \frac{(n-1)!n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}
\end{aligned}$$

Question 13: *How can we prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ without all this awful factorial nonsense?*

Answer 13: *We will count a particular set in two different ways! Let $X = \{1, 2, 3, \ldots, n\}$. Then, $\binom{n}{k}$ clearly counts the k-element subsets of X.*

We hope that we can somehow use $\binom{n-1}{k} + \binom{n-1}{k-1}$ to count the same thing, and do so by artificially introducing an additive-principle decomposition of the k-element subsets of X into two cases. Let $X' = \{1, 2, 3, \ldots, n-1\}$. Then, in constructing a subset S of X, we have two distinct possibilities:

Case I: S contains n. *Then, since we know one element of S, we only need to choose $k-1$ others to bring it up to $k-1$ elements. These elements must be numbers less than n; that is, elements of X'. There are thus $\binom{n-1}{k-1}$ possible such k-element sets.*

Case I: S does not contain n. *Since we haven't assigned any elements to S, we need to choose all k of them. These elements must be numbers less than n, since n was specifically excluded. There are thus $\binom{n-1}{k}$ possible such k-element sets.*

Using the additive principle to assemble these, we see that there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ subsets of X with k elements.

Since the binomial coefficient has such a simple enumerative interpretation, it lends itself to many equivalencies of enumerations:

Question 14: *Prove that*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Answer 14: *The left side of this equation is easily interpreted as a count related to subsets of a set with n elements. Let $X = \{1, 2, \ldots, n\}$. Then, the terms of the sum on the left side count the number of zero-element subsets of X, then the number of one-element subsets, then the number of two-element subsets, and so forth up to the number of n-element subsets. Adding together the number of subsets of each size gives a number with a simple interpretation: it is the number of subsets of X, regardless of size.*

Now, to form an equality with the right side of the equaiton, we want to find a way to argue that the nubmer of subsets of X is also 2^n. To do so, we might consider the following construction technique for a subset of X: for each element of X, we have two choices; it is either in the subset or it is out of the subset. Since we make a choice among two alternatives and repeat this choice n times, the number of ways to build a subset is $2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 2^n$.

Almost all identities involving the combination statistic are amenable to a combinatorial proof, and frequently these proofs are much easier than trying to prove the same thing algebraically.

4 The Twelvefold Way

The "twelvefold way" is a term developed by Richard Stanley to describe 12 closely related enumeration statistics; in the course of developing these statistics, we'll learn a great deal about enumerative methods.

We've seen several different sorts of ways to enumerate "selections" so far. There were our strings, where we cared about order but didn't force any relationship among the string elements; there were multisets, where we allowed repetitions but didn't care about order, there were strings of distinct elements, where we cared about order and were restricting choices, and there were sets, which didn't regard order but had choice-restriciton. It's clear that the question "how many different ways can we take n things from a pool of k?" has more than one interpretation. In order to specify this question, we need to know more about what distinguishes two "ways", and what restrictions we have on our choice. There are three properties of a selection statistic which we'll find of interest:

Selected-element restriction Can we select the same element multiple times? Do we *demand* that each element be selected at least once?

Order consideration Is the choice "AAC" different from "ACA"?

Element consideration Do the individual things chosen have intrinsic properties? Do we think of "ABC" merely as "choosing 3 different things" and indistinguishable from "CAB" or even "XYZ"? We would always think of "AAB" and "ABC" as different, however; one has a repetition, and the other does not.

All in all, we will have 12 different statistics, based on our answer to these questions:

Choosing k el'ts from a set of size n	Free choice	No repeats	Each el't at least once
Ordered choice, distinguished el'ts	(1)	(2)	(3)
Unordered choice, distinguished el'ts	(4)	(5)	(6)
Ordered choice, undistinguished el'ts	(7)	(8)	(9)
Unordered choice, undistinguished el'ts	(10)	(11)	(12)

We've actually seen examples of (1), (2), and (5) already.

There are two common other ways of interpreting these enumerative statistics, which may be useful, or easier to visualize.

Gian-Carlo Rota presented these statistics as a question of putting "balls into boxes". For instance, instead of selecting the three-element subset $\{1, 3, 7\}$ from $\{1, 2, \ldots, 10\}$, he would visualize this as a system of 10 numbered boxes, with three unlabeled balls placed in boxes 1, 3, and 7. In contrast, if he wanted to describe the sequence $(1, 7, 3)$, he would describe the same set of boxes, but numbered balls to indicate order of appearance, so that ball 1 would go in box 1, ball 2 in box 7, and ball 3 in box 3. Thus, the same statistics could be represented by the number of configurations of balls into boxes, depending on box capacity and whether things were labeled.

Putting k balls into n boxes	Free choice	≤ 1 ball per box	≥ 1 ball per box
Labeled balls, labeled boxes	(1)	(2)	(3)
Unlabeled balls, labeled boxes	(4)	(5)	(6)
Labeled balls, unlabeled boxes	(7)	(8)	(9)
Unlabeled balls, unlabeled boxes	(10)	(11)	(12)

Another popular interpretation of these statistics is that of Richard Stanley, where he counts equivalence classes of functions from a set of size k to a set of size n. The question of equivalence depends on whether two functions are identical up to permutation of the elements of one set; e.g. If $X = \{A, B, C\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the function $\{(A, 2), (B, 5), (C, 8)\}$ would be considered as equivalent to $\{(A, 8), (B, 2), (C, 5)\}$ under a permutation of the elements of X.

This corresponds to order-distinguishability, or ball-labeling in the two prior interpretations. In this interpretation, the restrictions on repeated choice or requirement to choose all elements can easily be stated as injectivity and surjectivity:

Functions from X to Y with $\|X\| = k$, $\|Y\| = n$	Any functions	Injective functions	Surjective functions
In total	(1)	(2)	(3)
Equivalence classes under permuting X	(4)	(5)	(6)
Equivalence classes under permuting Y	(7)	(8)	(9)
Equivalence classes under permuting $X$and Y	(10)	(11)	(12)

Now, with many interpretations of these statistics, we are in a position to start discussing the actual values in each cell.

Cell (5) counts the number of unordered choices of k distinct elements from an n-element set. This we know to be $\frac{n!}{k!(n-k)!} = \binom{n}{k}$.

Cell (2) counts the number of ordered choices of distinct elements; this is actually a precursor to our calculation for cell (5), which we attained via symmetry-division from the number of ordered choices. The number of ordered choices was $n(n-1)(n-2)(n-3)\cdots(n-k+1) = \frac{n!}{(n-k)!} = k!\binom{n}{k}$. This is sometimes known as the *permutation statistic*, and may be denoted ${}_nP_k$.

Cell (1) is also easy to count using the multiplicative principle. We choose any of n values for our first choice; we choose any of n values for our second choice, and so forth k times over, which we can do in $n \cdot n \cdot n \cdot n \cdots n = n^k$ ways.

The next easy cell to fill is the surprisingly esoteric (11) and almost incidentally (8). In the balls-and-boxes paradigm, this would be the question: how many ways can we put k balls in n unlabeled boxes such that each box contains no more than one ball? If $k > n$, this is clearly impossible. If $k \leq n$, then we could put one ball per box until we have k boxes containing one ball each, and $n-k$ boxes containing no balls. If the boxes are unlabeled, all ways of doing this are identical. Thus, statistics (8) and (11) happen to always be either 1 or 0, depending $n \geq k$.

We thous have a pretty good start on filling out our table:

Choosing k el'ts from a set of size n	Free choice	No repeats	Each el't at least once
Ordered choice, distinguished el'ts	n^k	$k!\binom{n}{k} = {}_nP_k$	?
Unordered choice, distinguished el'ts	?	$\binom{n}{k}$	?
Ordered choice, undistinguished el'ts	?	$\begin{cases} 1 \text{ if } n \geq k \\ 0 \text{ if } n < k \end{cases}$	?
Unordered choice, undistinguished el'ts	?	$\begin{cases} 1 \text{ if } n \geq k \\ 0 \text{ if } n < k \end{cases}$	?

4.1 Cells (4) and (6): quantity assignments

Cells (4) and (6) of the twelvefold way can easily be identified as assignments of quantities to each of n different entities. This is most evident in the balls-in-boxes paradigm, under which we are simply distributing k balls among n boxes, freely in the case of cell (4), freely after assigning one to each box in the case of cell (6). We might ask this question for specific values of n and k to get a better idea of what we're looking at:

Question 15: *How many ways are there to distribute* 10 *identical items (e.g. coins) among four individuals so that each individual receives at least one item?*

Answer 15: *We culd list all the ways to do this out, but it would be astonishingly tedious to do so. A revisualization of the problem, though, allows us to rephrase it in terms of a quantity we know how to find. A division of 10 coins among 4 individuals could be represented by laying out all 10 coins in a row, and then placing 3 dividers, and assigning each section to an individual. So, for instance, we could represent a division in which Alice receives 3 coins, Bob 4, Carl 2, and Diane 1 with the following placement of dividers:*

$$\circ\ \circ\ \circ \,|\, \circ\ \circ\ \ \circ\ \circ \,|\, \circ\ \circ \,|\, \circ$$

So, in general, we can associate a division of 10 items among people with a choice of three divider-positions among the 10 items. Since the dividers go between the items, there are 9 possible divider-locations among 10 items; as a result, there are $\binom{9}{3} = 84$ *ways to do so.*

Generalizing the above, we see that if we are assigning k items to n individuals such that each individual gets at least one, we can view this as placing $n-1$ dividers in the $k-1$ interstices between the items, and that there are $\binom{k-1}{n-1}$ ways to do so.

We can use a similar approach to find a statistic for cell (4): we can either bootstrap off of our known value for (6), or subtly modify it. To bootstrap, note that we can form an easy bijection between free asignments of k items to n entities and assignments of $n+k$ items to n entities in which each entity gets at least one item. The bijection is simple: distribute the n extra items giving one to each individual or reverse the process by taking one item away from each individual. Thus, all we need to know is how to calculate statistic (6) with parameters n and $n+k$: this would give $\binom{n+k-1}{n-1}$.

Alternatively, we could find (4) by simply considering the placement of items and dividers as was done for (6), but here placing multiple dividers adjacent is not a problem, so instead of fixing the location of n items and putting dividers among them, we consider $n+k-1$ open spaces, and fill each with either one of k items item or one of $n-1$ dividers. We are thus selecting $n-1$ locations from $n+k-1$ possible lcoations for our dividers, which can be done in $\binom{n+k-1}{n-1}$ ways.

Our table for the twelvefold way is rapidly filling!

Choosing k el'ts from a set of size n	Free choice	No repeats	Each el't at least once
Ordered choice, distinguished el'ts	n^k	$k!\binom{n}{k}$	?
Unordered choice, distinguished el'ts	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Ordered choice, undistinguished el'ts	?	$\begin{cases} 1 \text{ if } n \geq k \\ 0 \text{ if } n < k \end{cases}$	?
Unordered choice, undistinguished el'ts	?	$\begin{cases} 1 \text{ if } n \geq k \\ 0 \text{ if } n < k \end{cases}$	?

The five remaining spaces will require more clever tools to fill in.